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The Instability of the Nash Equilibrium in Common-Pool Resources

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Abstract

Efficient allocations in common-pool resources cannot be accomplished when appropriators are selfish. In addition to this dilemma, using a linear approximation of best response functions at the Nash equilibrium in the common-pool resource environment, we find that the system of simultaneous difference equations is locally *unstable* if the number of appropriators is at least four. This result indicates that the equilibrium analysis might not capture the essence of the common-pool resource problem, but provides an answer to "some unexplained pulsing behavior" (Ostrom, *Journal of Economic Behavior & Organization*, vol. 61, no.2 [2006], p. 150) of each appropriator's labor input in common-pool resource experiments.

JEL classification codes: C62, C72, C92, Q22

1. Introduction

An open access resource such as a fishing ground, an irrigation system, or a forest is called a *common-pool resource* (CPR)[1]. Since a CPR is a private good, once an appropriator catches fish, it is no longer available for others. In such a situation, each appropriator has a strong incentive to obtain more and more units. This might eventually lead to overexploitation or destruction of the CPR, which is called "the tragedy of the commons" (Hardin, 1968). The simplest form of the analysis is to understand the *static* appropriation dilemma using the Nash equilibrium concept in which the labor input of each appropriator is a strategy[2]. Using the Nash equilibrium labor inputs for production results in overexploitation. This standard equilibrium analysis implicitly assumes that the equilibrium is stable. However, what if it is not stable?

In an overview of experimental studies on behavior and outcomes in common-pool resource (CPR) dilemmas, Ostrom (2006) summarized previous experimental results, including those of Walker, Gardner, and Ostrom (1994), as follows:

> The average payoffs achieved in the baseline experiments did approach the Nash equilibrium of the non-cooperative game, but with some *unexplained* (italics added) pulsing behavior….
> … At the individual level, however, subjects rarely invested the predicted level of investment at equilibrium. Instead, all experiments provided evidence of a strong pulsing pattern in which individuals appeared to increase their investments in the common-pool resource until there was a strong reduction in yield, at which time they tended to reduce their investments leading to an increase in yields. The pattern was repeated over time. (pp. 150, 153)

This paper provides a theoretical basis for the pulsing behavior of appropriators' labor input in a CPR. Using a linear approximation of best response functions at the Nash equilibrium, we show that with at least four appropriators the system of simultaneous difference equations is locally unstable. In other words, the system becomes *unstable* if each appropriator follows the best response to the sum of other appropriators' previous labor

[1] See Ostrom (2006) for the definition.

[2] Ostrom, Gardner, and Walker (1994) classified CPR problems into two types: appropriation problems and provision problems. "In appropriation problems, we focus attention on the *flow* aspect of the CPR. In provision problems, we concentrate on the *facility* aspect of the CPR" (p. 105). In this paper, we focus on appropriation problems.

inputs. We obtain the instability result *without* considering the dynamic adjustment of stock of resources[3]. That is, both instability and inefficiency are *intrinsic* properties of common-pool resources.

Thus far, researchers have considered that there is a CPR dilemma between the Pareto efficient and Nash allocations. However, in addition to the dilemma, each appropriator might not be able to benefit even from a Nash equilibrium allocation because of system instability even in a static setting. In other words, an equilibrium analysis might be inappropriate. That is, there is a CPR "trilemma" among these allocations; hence, the problem should be considered much more difficult than earlier believed.

Section 2 provides the proof of instability of the Nash equilibrium. Section 3 summarizes the case for simultaneous differential equations. Section 4 briefly describes our findings and discusses further research.

2. The Difference Equations Case

A society has n (≥2) appropriators, each of whom faces a decision problem of how to divide her endowment between catching fish and personal leisure time. Let w_i be appropriator i's initial endowment, which is the total possible leisure time, and x_i be the labor input for catching fish. We assume that the production function is concave: let it be $f(x)$ with $f(0) = 0$, $f(x) > 0$ if $x > 0$ and $f''(x) < 0$, where $x = \sum x_j$.[4] Then, the average product is always greater than the marginal product, that is, $f'(x) < f(x)/x$.[5] The amount of fish that appropriator i catches is proportional to the time spent on catching fish with respect to the total sum of all appropriators' labor inputs. We normalize the price of fish as 1 and denote the wage rate by p. Then, appropriator i's income is defined by

$$m_i(x_i, x_{-i}) = f(x_i + x_{-i})\frac{x_i}{x_i + x_{-i}} + p(w_i - x_i),$$

where $x_{-i} = \sum_{i \neq j} x_j$.[6] The optimization problem of appropriator i is

$$\max_{x_i} m_i(x_i, x_{-i}) \text{ subject to } x_i \in [0, w_i].$$

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[3] For example, the main focus of standard graduate textbooks such as *Environmental Economics* by Hanley, Shogren, and White (2007) is to understand the dynamics of the flow and stock of resources. Hanley et al. (2007) describe the inefficiency problem in a static setting using the Nash equilibrium, but not the stability (see pp. 271-2).

[4] $f'(x) > 0$ is not necessary in the following argument.

[5] If $f''(x) < 0$, $f'(x)x - f(x) < 0$ follows immediately.

[6] This is a standard setup for CPR. See Ostrom (2006).

A list of labor inputs $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$ is a *Nash equilibrium* if, for all *i*,

$$m_i(\hat{x}_i, \hat{x}_{-i}) \geq m_i(x_i, \hat{x}_{-i}) \text{ for all } x_i \in [0, w_i].$$

Define

$$r_i(x_{-i}) = \underset{x_i}{\text{argmax}} \left\{ m_i(x_i, x_{-i}) \middle| x_i \in [0, w_i] \right\}$$

which is the *best response function* of appropriator *i*.

At time *t*, let appropriator *i*'s decision of labor input be x_i^t. Assume that appropriator *i* chooses $r_i(x_{-i}^t)$ at time *t* + 1, where $t = 1,2,\ldots$ Then, the system

$$x_i^{t+1} = r_i(x_{-i}^t) \ (i = 1,2,\cdots,n) \tag{1}$$

is *locally stable* at Nash equilibrium $\hat{\mathbf{x}}$ if the system $x_i^{t+1} = \frac{dr_i(\hat{x}_{-i})}{dx_{-i}} x_{-i}^t + k_i$ $(i = 1,2,\cdots,n)$ is stable, where $k_i = \hat{x}_i - \frac{dr_i(\hat{x}_{-i})}{dx_{-i}} \hat{x}_{-i}$. Note that the system $x_i^{t+1} = \frac{dr_i(\hat{x}_{-i})}{dx_{-i}} x_i^t + k_i$ $(i = 1,2,\cdots,n)$ is the linear approximation of the system $x_i^{t+1} = r_i(x_{-i}^t)$ $(i = 1,2,\cdots,n)$ at Nash equilibrium $\hat{\mathbf{x}}$. In order to simplify the stability argument of the system of simultaneous difference equations, the following well-known property is useful.

Property 1.[7] (i) *Each appropriator's best response function is independent of her endowment, that is,* $r_i = r_j$ *for all i and j.*

(ii) *There is no asymmetric Nash equilibrium, that is,* $\hat{x}_i = \hat{x}_j$ *for all i and j at the Nash equilibrium.*

Proof. (i) Define $M_i(x_i, x_{-i}) = \frac{\partial m(x_i, x_{-i})}{\partial x_i}$. Since the best response function is a relation between x_{-i} and x_i with the first-order condition of the maximization, i.e., $M_i(x_i, x_{-i}) = 0$, consider

$$M_i(x_i, x_{-i}) = \frac{1}{(x_i + x_{-i})^2} \{((f'(x_i + x_{-i})x_i + f((x_i + x_{-i}))(x_i + x_{-i}) - f(x_i + x_{-i})x_i\} - p = 0.$$

Then, this equation is independent of w_i and the relation between x_{-i} and x_i is the same for all appropriators. That is, $r_i = r_j$ for all *i* and *j*.

(ii) Since the sum of the first-order conditions of all appropriators should be zero at a Nash equilibrium, we have

[7] See, for example, Walker, Gardner, and Ostrom (1990) and Ito, Saijo, and Une (1995).

$$\sum M_i(\hat{x}_i,\hat{x}_{-i}) = \sum\left\{\frac{1}{\hat{x}^2}\left\{\left(f'(\hat{x})\hat{x}_i + f(\hat{x})\right)\hat{x} - f(\hat{x})\hat{x}_i\right\} - p\right\}$$
$$= \frac{1}{\hat{x}^2}\left\{f'(\hat{x})\hat{x}^2 + nf(\hat{x})\hat{x} - f(\hat{x})\hat{x}\right\} - np$$
$$= f'(\hat{x}) + (n-1)\frac{f(\hat{x})}{\hat{x}} - np = 0,$$

where $\hat{x} = \sum \hat{x}_i$. Therefore,

$$f'(\hat{x}) - \frac{f(\hat{x})}{\hat{x}} = -n\left\{\frac{f(\hat{x})}{\hat{x}} - p\right\} < 0 \ (\because f'(x)x - f(x) < 0).$$

Since

$$M_i(\hat{x}_i, \hat{x}_{-i}) = \frac{\hat{x}_i}{\hat{x}}\left\{f'(\hat{x}) - \frac{f(\hat{x})}{\hat{x}}\right\} + \frac{f(\hat{x})}{\hat{x}} - p = \left\{1 - n\frac{\hat{x}_i}{\hat{x}}\right\}\left\{\frac{f(\hat{x})}{\hat{x}} - p\right\} = 0,$$

we obtain $\hat{x}_i = \frac{1}{n}\hat{x}$ for all *i*. ■

Since $r_i = r_j$, let $r = r_i = r_j$ and $k = k_i$. Define $a \equiv \frac{dr(\hat{x}_{-i})}{dx_{-i}}$. The following property shows that the best response function is downward sloping.

Property 2. *The slope of the best response function r at the Nash equilibrium* $\hat{\mathbf{x}}$ *is negative, that is,* $a < 0$.

Proof. Consider appropriator *i*. Totally differentiating the first-order condition $M_i(x_i, x_{-i}) = 0$, we have $\frac{\partial M_i(x_i,x_{-i})}{\partial x_i}dx_i + \frac{\partial M_i(x_i,x_{-i})}{\partial x_{-i}}dx_{-i} = 0$. Then, the slope of the best response function of appropriator *i* is

$$\frac{dx_i}{dx_{-i}} = a = -\frac{\frac{\partial M_i(x_i,x_{-i})}{\partial x_{-i}}}{\frac{\partial M_i(x_i,x_{-i})}{\partial x_i}}, \text{ where}$$

$$\frac{\partial M_i(x_i,x_{-i})}{\partial x_{-i}} = \frac{x_i}{x_i + x_{-i}}f''(x_i + x_{-i}) + \left\{f'(x_i + x_{-i}) - \frac{f(x_i + x_{-i})}{x_i + x_{-i}}\right\}\frac{x_{-i} - x_i}{(x_i + x_{-i})^2} \text{ and}$$

$$\frac{\partial M_i(x_i,x_{-i})}{\partial x_i} = \frac{x_i}{x_i + x_{-i}}f''(x_i + x_{-i}) + \left\{f'(x_i + x_{-i}) - \frac{f(x_i + x_{-i})}{x_i + x_{-i}}\right\}\frac{2x_{-i}}{(x_i + x_{-i})^2}.$$

Since $\hat{x}_{-i} = (n-1)\hat{x}_i$ by Property 1, we obtain

$$\frac{dx_i}{dx_{-i}} = -\frac{f''(\hat{x}_i+\hat{x}_{-i}) + \left\{ f'(\hat{x}_i+\hat{x}_{-i}) - \frac{f(\hat{x}_i+\hat{x}_{-i})}{\hat{x}_i+\hat{x}_{-i}} \right\} \frac{n-2}{n\hat{x}_i}}{f''(\hat{x}_i+\hat{x}_{-i}) + \left\{ f'(\hat{x}_i+\hat{x}_{-i}) - \frac{f(\hat{x}_i+\hat{x}_{-i})}{\hat{x}_i+\hat{x}_{-i}} \right\} \frac{2(n-1)}{n\hat{x}_i}} \equiv -\frac{A}{B}. \tag{2}$$

Since $f'' < 0$ and $f'(x)x - f(x) < 0$, A and B are negative. Hence, $\frac{dr(\hat{x}_{-i})}{dx_{-i}} = -\frac{A}{B} < 0.$ ∎

Applying the standard results in non-negative matrices and stability to our setup, we obtain a necessary and sufficient condition for local stability.

Property 3. *The system* $x_i^{t+1} = r_i(x_{-i}^t)$ *(*$i = 1,2,\cdots,n$*) is locally stable if and only if* $|a(n-1)| < 1$.

Proof. System (1) can be written as

$$\begin{bmatrix} x_1^{t+1} \\ \vdots \\ \\ x_n^{t+1} \end{bmatrix} = a \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^t \\ \vdots \\ \\ x_n^{t+1} \end{bmatrix} + \begin{bmatrix} k \\ \vdots \\ \\ k \end{bmatrix}.$$

Alternatively, it can be written as $\mathbf{x}^{t+1} = \mathbf{A}\mathbf{x}^t + \mathbf{k}$, where

$$\mathbf{A} = a(n-1) \begin{bmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & & \vdots \\ \vdots & & \ddots & \frac{1}{n-1} \\ \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{bmatrix}, \mathbf{x}^t = \begin{bmatrix} x_1^t \\ \vdots \\ x_n^t \end{bmatrix} \text{ and } \mathbf{k} = \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix}.$$

Let $\mathbf{A} = a(n-1)\mathbf{B}$. Then, we obtain

$\mathbf{x}^t = \mathbf{A}^{t-1}\mathbf{x}^1 + [\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{t-2}]\mathbf{k}$ or

$\mathbf{x}^t = a^{t-1}(n-1)^{t-1}\mathbf{B}^{t-1}\mathbf{x}^1 + [\mathbf{I} + a(n-1)\mathbf{B} + \cdots + a^{t-2}(n-1)^{t-2}\mathbf{B}^{t-2}]\mathbf{k}$.

First, note that $\mathbf{B}^t = [b_{ij}^t]$ is a probability matrix (i.e., $\sum_j b_{ij}^t = 1$ for all *j* and $0 \le b_{ij}^t \le 1$) since matrix $\mathbf{B}$ is a probability matrix. Hence, $0 \le b_{ij}^t \le 1$ for all *i*, *j*, and *t*. Therefore, if $|a(n-1)| < 1$, then $\lim_{t\to\infty}(a(n-1)\mathbf{B})^t = \mathbf{0}$. Second, since $\mathbf{B}$ is a probability matrix, the Frobenius root of $\mathbf{B}$, $\lambda(\mathbf{B})$, is 1 (see Theorem 20.3 in Nikaido [1970]). Moreover, the series

$[\mathbf{I}+a(n-1)\mathbf{B}+\cdots+a^t(n-1)^t\mathbf{B}^t+\cdots]$ converges if and only if $|a(n-1)|<1=\lambda(\mathbf{B}^2)$. Since a < 0 by Property 2 and

$$[\mathbf{I}+a(n-1)\mathbf{B}+\cdots+a^t(n-1)^t\mathbf{B}^t+\cdots] = [\mathbf{I}+a^2(n-1)^2\mathbf{B}^2+\cdots+a^{2t}(n-1)^{2t}\mathbf{B}^{2t}+\cdots]$$
$$-[-a(n-1)\mathbf{B}-\cdots-a^{2t+1}(n-1)^{2t+1}\mathbf{B}^{2t+1}-\cdots], \quad (3)$$

each term on the right-hand side of (3) converges if and only if $|a(n-1)|<1=\lambda(\mathbf{B}^2)$ applying Theorem 19.1 in Nikaido (1970). Therefore, system (1) is locally stable if and only if $|a(n-1)|<1$. ■

As the proof of Property 3 shows, the local stability property does not depend on the initial value x^1. The following proposition shows that the best response dynamics is always unstable as long as the number of appropriators is at least four.

Proposition 1. *If* $n \geq 4$*, then the system* $x_i^{t+1} = r_i(x_{-i}^t)$ *(*i *= 1,2,⋯,*n*) is locally unstable at the Nash equilibrium* $\hat{\mathbf{x}}$.

Proof. Since each term of A and B is negative in (2),

$$1-|a(n-1)| = 1-\frac{A}{B}(n-1) = \frac{1}{B}(B-A(n-1))$$
$$= \frac{1}{B}\left\{f''(\hat{x})+\left(f'(\hat{x})-\frac{f(\hat{x})}{\hat{x}}\right)\frac{2(n-1)}{n\hat{x}_i}-(n-1)f''(\hat{x})-\left(f'(\hat{x})-\frac{f(\hat{x})}{\hat{x}}\right)\frac{(n-1)(n-2)}{n\hat{x}_i}\right\}$$
$$= \frac{-1}{B}\left\{f''(\hat{x})(n-2)+\frac{1}{n\hat{x}_i}\left(f'(\hat{x})-\frac{f(\hat{x})}{\hat{x}}\right)(n-1)(n-4)\right\}.$$

Since B < 0, $f''(\hat{x})<0$ and $f'(\hat{x})\hat{x}-f(\hat{x})<0$, $1<|a(n-1)|$ if $n \geq 4$. ■

Let us consider the implications of Proposition 1. First, the result does not depend on the shape of the production function with very mild conditions such as $f(0) = 0$, $f(x) > 0$ if $x > 0$ and $f''(x) < 0$, or the allocation of endowment. In this sense, the instability result is *intrinsic* to CPR problems.

Second, the number of appropriators is important for CPR analysis. As Ostrom (2006) indicated, “We also wanted to include a sufficient number of players whose knowledge of outcomes did not automatically provide information about each player’s actions, again similar to the field” (p.152). The number of appropriators in Walker, Gardner, and Ostrom (1990) is eight. In this sense, we found the minimum number of appropriators sufficient to capture the essence of CPR instability.

Consider an example when $f(x) = 7.45\sqrt{x}$ in Figure 1. The horizontal axis represents the sum of the other appropriators' labor input, and the vertical axis represents the labor input of appropriator *i*'s response to x_{-i}. Note that all appropriators have the same best response function $r(x_{-i})$. Let w = 20 for all appropriators. Consider the case of three appropriators, that is, the left half of the box. The slope of line 0*t* is 1/2; hence, the Nash equilibrium must be the intersection of line 0*t* and $r(x_{-i})$, at *E*. Suppose that every appropriator's input is the same at *s*. Now, the best response is *a*. Then, the next point should be *b*, and the next-best response is *c*. This process continues until *E* is reached, which is the Nash equilibrium when n = 3.

Consider a case of five appropriators, that is, the entire box. The slope of line 0*f* is 1/4; hence, the Nash equilibrium must be the intersection of line 0*f* and $r(x_{-i})$, which is *E′*. Suppose that every appropriator's input is the same at *s′*. Now, the best response is *a*. Then, the next point should be *b′* and the best response is *c′*. Thereafter, the response should be 0-*g*-*h*-*k*. This cycle continues without reaching *E′*, which is the Nash equilibrium when n = 5. This could be a source of pulsing behavior.

Consider the 0-*g*-*h*-*k* cycle. When every appropriator chooses zero labor input, *k*, the number of fish from the CPR is zero and hence the benefit from fishing is zero. When every appropriator chooses *g*, then this results in the labor input more than the Nash equilibrium labor input that is the height of *E′*. In addition to this, when every appropriator chooses *k*, the labor input is zero, and hence t That is, the efficiency of the 0-*g*-*h*-*k* cycle is worse than that of the Nash equilibrium.

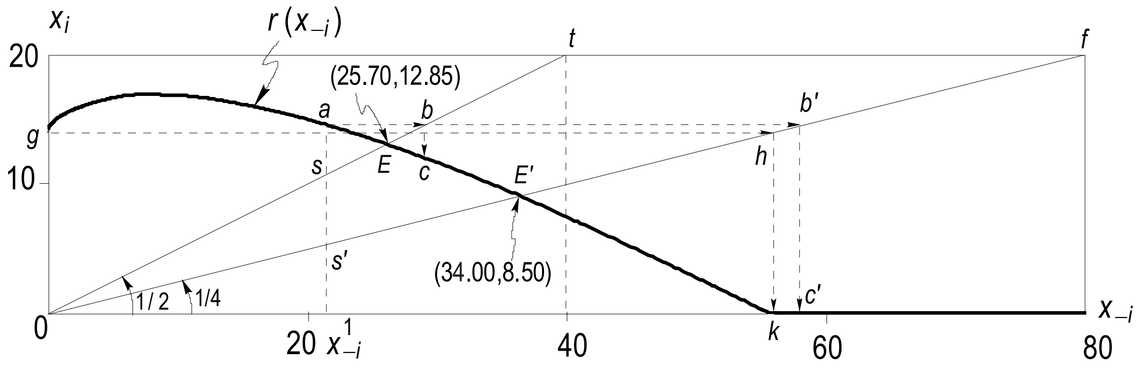

Figure 1. Stability properties when $f(x) = 7.45\sqrt{x}$, p = 1 and w = 20.

As Figure 1 shows, the best response function is independent of the number of appropriators. That is, as the number of appropriators increases from three to five, for example, the size of the box expands to the right, and hence, the jump from *a* should be amplified from *b* to *b′*. This is why the number of appropriators plays an important role in the stability of the system of difference equations in CPR.

3. The Differential Equations Case

Using the linear approximation of the best response functions, we consider the system of simultaneous differential equations as follows.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k} - \mathbf{x}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{x}(t) + \mathbf{k} \qquad (4)$$

The dynamic property of (4) is governed by the eigenvalues $\lambda(\mathbf{A} - \mathbf{I})$. Simple computation yields the following:[8]

$$\lambda(\mathbf{A} - \mathbf{I}) = ((n-1)a - 1, \underbrace{-a-1, -a-1, \ldots, -a-1}_{n-1})$$

Since $(n - 1)a - 1 < 0$, the system is asymptotically stable if $a < -1$. That is,

Proposition 2. *If* $a > -1$, *then the system* $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k} - \mathbf{x}(t)$ *is asymptotically stable at the Nash equilibrium* $\hat{\mathbf{x}}$.

That is, the stability property of the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k} - \mathbf{x}(t)$ does not depend on the number of appropriators, but only on the slope of the best response function at the Nash equilibrium. Furthermore, the system $\mathbf{x}^{t+1} = \mathbf{A}\mathbf{x}^{t} + \mathbf{k}$ is always unstable with at least four appropriators and the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{k} - \mathbf{x}(t)$ is stable with $a > -1$. That is, the experimental evidence of pulsing behavior provided by Ostrom (2006) supports $\mathbf{x}^{t+1} = \mathbf{A}\mathbf{x}^{t} + \mathbf{k}$.

4. Concluding Remarks

In this article, we showed that a system of simultaneous difference equations with four or more appropriators would be locally unstable. In the case of simultaneous differential equations, the system would be stable if the slope of the best response function at the Nash equilibrium is greater than -1. These results indicate that the equilibrium

[8] See the appendix.

analysis might not capture the essence of the common-pool resource problem whenever the system is not stable. Furthermore, the former result provides an answer to "some unexplained pulsing behavior" of each appropriator's labor input in common-pool resource experiments summarized by Ostrom (2006).

Walker, Gardner, and Ostrom (1990), Walker and Gardner (1992), Ostrom, Gardner, and Walker (1994), and Casari and Plott (2003) all used experimental designs with n = 8 and a = -1/2, and hence the system is unstable in the difference equation system. Although individual data are not provided in the papers, all studies seemingly share the pulsing behavior. On the other hand, Cason and Gangadharan (2014) used an experimental design with n = 4 and a = -1/2, and hence $|a(n-1)| = 3/2 > 1$, which shows the system is unstable.[9] However, the standard errors of the individual data are quite low, and they reported that peer punishment works basically well. This suggests that subjects might use some other principles of dynamic behavior and/or institutional devices such as the effect of punishment on behavior. These are new open questions.

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[9] Hayo and Vollan (2012) used n = 5 and a = -1/2, but it is uncertain that they found pulsing behavior. Rodriguez-Sickert, Guzmán, andCárdenas (2008) used n = 5, but each player has the dominant strategy in their design.

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Appendix

Property A1: $\lambda(\mathbf{A}-\mathbf{I}) = ((n-1)a-1, \underbrace{-a-1, -a-1, \ldots, -a-1}_{n-1})$.

Proof. The characteristic equation of the differential equation is given by

$$|\mathbf{A}-\mathbf{I}-\lambda\mathbf{I}| = \begin{vmatrix} -1-\lambda & a & \cdots & a \\ a & -1-\lambda & & \vdots \\ \vdots & & & a \\ a & \cdots & a & -1-\lambda \end{vmatrix} = 0$$

This determinant can be simplified through some elementary matrix operations, as follows. Subtracting the first row from all other rows yields

$$|\mathbf{A}-\mathbf{I}-\lambda\mathbf{I}| = \begin{vmatrix} -1-\lambda & a & \cdots & a \\ a+1+\lambda & -(a+1+\lambda) & & \vdots \\ \vdots & & & 0 \\ a+1+\lambda & \cdots & 0 & -(a+1+\lambda) \end{vmatrix} = (-a-1-\lambda)^{n-1} \begin{vmatrix} -1-\lambda & a & \cdots & a \\ -1 & 1 & & \vdots \\ \vdots & & & 0 \\ -1 & \cdots & 0 & 1 \end{vmatrix}$$

Further, subtracting from the first row all the other rows multiplied by *a* yields

$$|\mathbf{A}-\mathbf{I}-\lambda\mathbf{I}| = (-a-1-\lambda)^{n-1} \begin{vmatrix} -1-\lambda+(n-1)a & 0 & \cdots & 0 \\ -1 & 1 & & \vdots \\ \vdots & & & 0 \\ -1 & \cdots & 0 & 1 \end{vmatrix} = (-a-1-\lambda)^{n-1}(-1+(n-1)a-\lambda)$$

Thus, we have $\lambda(\mathbf{A}-\mathbf{I}) = ((n-1)a-1, \underbrace{-a-1, -a-1, \ldots, -a-1}_{n-1})$. ∎