KOCHI UNIVERSITY OF TECHNOLOGY

Social Design Engineering Series *SDES-2023-4*

The general equilibrium effects of localised technological progress: A Classical approach

Naoki Yoshihara
Department of Economics,University of Massachusetts, Amherst
Institute of Economic Research, Hitotsubashi University
School of Management, Kochi University of Technology

Roberto Veneziani
School of Economics and Finance, Queen Mary University of London

May 10, 2023

School of Economics and Management
Research Institute for Future Design
Kochi University of Technology

The general equilibrium effects of localised technological progress: A Classical approach*

Naoki Yoshihara† Roberto Veneziani‡

May 5, 2023

Abstract

We study the general equilibrium effects of localised technical progress à la Atkinson-Stiglitz in economies in which capital is a vector of reproducible and heterogeneous goods. We show that there is no obvious relation between ex-ante profitable innovations and the functional distribution of income that actually emerges in equilibrium. Unlike in the standard macroeconomic approach to technical progress, localised innovations may lead to indeterminacy in equilibrium factor prices, and individually rational choices of technique do not necessarily lead to optimal outcomes. Innovations may even cause the disappearance of *all* equilibria.

JEL: O33; D33; D51.

Keywords: localised technical progress, income distribution, general equilibrium.

*We would like to thank two referees and the Associated Editor for detailed and careful comments. We are grateful to Hiroyuki Sasaki, Reiner Franke, Kazuhiro Kurose, Neri Salvadori, Gil Skillman, Giacomo Corneo, Heinz Kurz, Woojin Lee, John Roemer, Yongsheng Xu, Nobusumi Sagara, Bertram Schefold, Rajiv Sethi, Peter Skott, and audiences at Southern Economic Association 2016, UMass Amherst, Tohoku University, Toyama University, Queen Mary University of London, Seoul National University, Waseda University, Nagoya University, and Kyoto University for insightful comments on an earlier draft. The usual disclaimer applies.

†(Corresponding author) Department of Economics, University of Massachusetts Amherst, Crotty Hall, Amherst, MA, 01002, USA; The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-0004, Japan; and School of Management, Kochi University of Technology, Tosayamada, Kami-city, Kochi 782-8502, Japan. (nyoshihara@econs.umass.edu)

‡School of Economics and Finance, Queen Mary University of London, Mile End Road, London E1 4NS, UK. (r.veneziani@qmul.ac.uk)

1 Introduction

In the last decade, the dynamics of profitability has been central in a number of debates, both in academia and in the popular press (such as those on widening inequalities and secular stagnation sparked, respectively, by Piketty [31] and Summers [40]). According to Summers ([40], p.66), the US economy has experienced "a substantial decline in the equilibrium or natural real rate of interest" in the last few decades and the "time series on the long-term real interest rate on a global basis, . . . shows a similar broad pattern of continuing decline" (ibid., p.71). A longer, historical perspective leads to similar conclusions: Figure 3.2 in Blanchard ([5], p.31) "shows a striking picture, with the safe rate decreasing from 10–15% in the 1300s to close to 0% today. It suggests a strong underlying negative trend of about 1.2 basis points . . . per year". The actual return on capital has also displayed a "decrease of about one-quarter to one fifth, from 4-5 percent in the eighteenth and nineteenth centuries to 3-4 percent today" (Piketty [31], p.206).[1]

While the long run dynamics of the (neutral or actual) rate of return is likely to be affected by many low frequency factors, including major demographic shifts, (see Blanchard [5], Chapter 3), one important question concerns the effect of innovations on the dynamics of profitability. For example, in the literature on secular stagnation, a key role is played by the slowdown of technical progress but also by its capital-saving nature (Summers [41], p.4). Similarly, the role played by technical progress was at the centre of the classical controversy on the so-called 'law of the tendential fall in the profit rate' with opposite views being expressed by Marx [26] and Schumpeter [38].

Yet, the standard macroeconomic analysis of technical progress provides only rather partial insights on the effects of innovations on profits, and on distribution, more generally. As Acemoglu ([2], p.443) has forcefully argued, the standard macroeconomic approach to innovations – the "orthodoxy" in Acemoglu's [2] words – has typically assumed that "technological improvements could be viewed as increasing productivity at all factor proportions (in particular, at all combinations of capital and labour)". This approach is "still widespread in much of macroeconomics" (ibid., p.456),[2] and it implies ignoring that technical change is typically *localised*, in that it "improves the productivity of the techniques (or 'activities') currently being used and perhaps some similar techniques with neighbouring capital-labour ratios" (ibid., p.443), and also *biased*, in that it has direct distributional implications – indeed, innovations are often introduced in order to economise on productive factors that become relatively scarce.

While there is now a substantial literature addressing biased, or directed, technical progress (see Acemoglu [2] and the contributions therein), localised technical change has received less attention. As Atkinson and Stiglitz ([3], p.573) noted in a classic paper, standard macroeconomics seems to "have forgotten the origins of the neo-classical production function: as the number of production processes increases (in an activity analysis model), the production possibilities can be more and more closely approximated by a smooth, differentiable curve. But the different points on the curve still represent different processes of production, and associated with each of these processes there will be certain technical knowledge specific to that technique . . . [I]f one brings about a technological improvement in one of the blue-prints this may have little or no effect on the other blue-prints" (ibid.).

In this paper, we follow Atkinson and Stiglitz [3] and focus on localised technical

[1] For a counterpoint, see Jordà et al [20].

[2] For a thorough discussion, see also Acemoglu ([1], section 2.7, chapters 15 and 20).

progress in an activity analysis model. Building on the classical framework developed by von Neumann [28], Sraffa [39] and Morishima [27], and later extended by Roemer [33, 34, 35, 37], we explore the effect of localised innovations on profitability and distribution in a fully disaggregated dynamic general equilibrium model.[3] Unlike in the standard macroeconomic literature (including the multi-sectoral extensions of the Ramsey model),[4] where capital is conceived as a single good, we model it as a bundle of reproducible and heterogeneous commodities. Apart from being more realistic, this assumption raises a number of interesting and complex issues, as is well known in capital theory (Sraffa [39]).

To be specific, we set up a dynamic general equilibrium model in which, in every period, agents exchange goods and services on a number of interrelated markets. Production takes time: capital and labour are traded at the beginning of each production period, while consumption goods are exchanged after production has taken place.

We adopt a Classical view of the functioning of capitalism and assume the economy to be driven by an accumulation motive.[5] A production technique is a blueprint describing how to combine a vector of produced inputs with labour in order to produce outputs. At the beginning of each period, the production set consists of a set of known blueprints and agents choose the activity that yields the maximum rate of profit. When innovations do emerge, they expand the production set by generating new blueprints that may be used at the beginning of the next production period.[6] The general equilibrium effect of this type of localised innovations is far from obvious.

Suppose the economy starts out on a balanced growth path with an equilibrium price vector and an optimally chosen technique that, absent any perturbations, would remain unchanged over time – the *status quo*. In the standard approach, technical progress is not localised, innovations lead to an inward shift of the isoquants and, under relatively mild conditions, the economy moves to a (unique) equilibrium in which the new technology is universally adopted, productive factors are fully employed, and a well defined functional distribution of income emerges, which depends on the bias of technical change.

A localised innovation, in contrast, is defined by the discovery of a new blueprint, that is, a single activity placed outside of the existing input-requirement set. Therefore, even if the original isoquants were smooth, localised technical progress introduces a kink in the new input-requirement set. Further, even if they expand the production possibilities set, not all localised innovations are necessarily cost-reducing.

We define localised innovations as profitable, when they are cost-reducing at status quo equilibrium prices, and prove that *if* (i) a new equilibrium exists in which (ii) a new profitable technique is adopted and (iii) the wage rate does not increase, then profits tend to increase (Theorem 2).[7] Once assumptions (i)-(iii) are relaxed, however, our findings

[3]To be precise, our results describe the effect of innovations on the key distributive variables, namely the real wage rate and the profit rate, and on the employment of productive factors. Thus, although we do not explicitly describe the dynamics of the income shares, our analysis does shed light on the effect of technical progress on the functional distribution of income.

[4]See, among the many others, Kongsamut et al. [22], Ngai and Pissarides [29], and Boppart [6]. For a critical survey see Kurose [23].

[5]For a detailed discussion and comparison of Classical and Neoclassical models and equilibrium concepts, see, for example, Roemer [35], Dana et al [9] and Kiedrowski [21].

[6]As in Atkinson and Stiglitz ([3]), we assume that localised innovations are the result of learning by doing, and do not explicitly consider R&D activities and the process of generating innovations. This allows us to focus on the general equilibrium effects of innovation on both prices, wages, and profits.

[7]Theorem 2 is a generalisation of the so-called 'Okishio theorem'. Okishio [30] proved that if the real wage rate is fixed at the (historically and culturally determined) subsistence level, then any cost-reducing

are much more nuanced and perhaps surprising.

First, if a new equilibrium with full employment of productive factors is reached after the generalised adoption of a profitable new activity, then the effect on distribution is a priori unclear, as there exist (infinitely) many combinations of profit rates and wage rates that can be supported in equilibrium. It is even possible for such a localised technical change not to be Pareto improving, as there are equilibria in which either the wage rate or the profit rate *decrease* compared to the equilibrium with the old technology (Theorem 3). The actual distributional outcome depends on the equilibrium selection mechanism.

This result is reminiscent of the well-known indeterminacy of the functional distribution of income in Sraffa's [39] system of equilibrium price equations (see, for example, Mandler [25] and Yoshihara and Kwak [46]). Nonetheless, the indeterminacy in Theorem 3 is quite different: it is the result of the equilibrium transition triggered by innovations, and it obtains under a more general equilibrium notion.

Second, more generally, the implications of technical progress depend on the general equilibrium effects of technical change on wages and profits. Localised innovations may contribute to maintain labour unemployed, or even – as conjectured by Acemoglu [2] – *create* technological unemployment in equilibrium. In these cases, technical change leads to an increase in the equilibrium profit rate (Theorem 4).

This conclusion does not hold in general, though, as cost-reducing localised innovations may lead the profit rate to *fall*. If the new technique increases labour productivity while it makes the capital stock abundant relative to the population, then its introduction drives the equilibrium profit rate to zero (Theorems 5 and 7). In line with Karl Marx's [26] famous intuition, an innovation that is individually profitable at status quo prices yields, after it is universally adopted, a change in the equilibrium price vector eventually leading the profit rate to decrease. Indeed, the equilibrium profit rate falls (albeit not necessarily to zero) even if the new technique worsens labour productivity, though in this case the mechanism is subtler and less intuitive, as the innovation is *not* adopted: it has a pure general equilibrium effect leading agents to opt for an older technique (Theorem 6).

Third, innovations can be highly disruptive and the process of 'creative destruction' is anything but smooth, as Schumpeter [38] emphasised. For there exist cost-reducing innovations that disrupt the status quo and yet are *not* adopted in the new equilibrium (Corollary 1). Innovations may paradoxically lead *older* techniques to become profitable again, due to changes in equilibrium prices (Theorem 6), or even lead to the disappearance of equilibrium altogether: the process of creative destruction entails disequilibrium dynamics (Proposition 2).

2 The economy

We analyse a generalisation of Roemer's [35, 37] model of an accumulating economy.

---

innovation will increase the equilibrium profit rate. This result has been interpreted as proving that the Marxian theory of the falling rate of profit is invalid, and sparked substantial controversy (see Roemer [33, 34] and Franke [12], and the references therein). In this vast literature, assumptions (i)-(iii) are typically made, and the general equilibrium effects of innovations are largely ignored.

2.1 Technology, innovation, and knowledge

Consider a closed economy with n produced goods. We focus on process innovations and assume the set of commodities to be constant over time. At the beginning of each production period $t = 1, 2, \ldots$, there is a finite set $\mathcal{B}_t$ of Leontief production techniques (A, L), where A is a $n \times n$ nonnegative, productive, and indecomposable matrix of material input coefficients, whose i-th column is denoted by A_i, and $L = (L_1, \ldots, L_i, \ldots, L_n) > \mathbf{0} \equiv (0, \ldots, 0)$ is a $1 \times n$ vector of labour coefficients.[8] The set $\mathcal{B}_t$ contains the blueprints that can be used at t to produce the n goods and the set of all conceivable production techniques at t, $\mathcal{P}_t$, is the convex hull of $\mathcal{B}_t$.[9]

The stock of knowledge does not depreciate: once a production technique is discovered, it remains available for agents to use. But knowledge can be accumulated. Formally, $\mathcal{B}_{t-1} \subseteq \mathcal{B}_t$ holds in general, and *technical progress* takes place between $t-1$ and t if and only if $\mathcal{B}_{t-1} \subset \mathcal{B}_t$ and $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$: (A^*, L^*) is an *innovation*, which is available in t.[10] Because we are interested in the effects of innovation on profitability in competitive market economies, we suppose that information both about $\mathcal{B}_{t-1}$ and about any new technique $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ is available to all agents in the economy in t.

At all t, for any $(A, L) \in \mathcal{B}_t$, let $\eta(A) = \frac{1}{1+\Pi(A)}$ denote the Frobenius eigenvalue of A. As A is nonnegative and productive, $\eta(A) < 1$ and $\Pi(A) > 0$.

2.2 Agents

We study some fundamental dynamic laws of capitalist economies characterised by a drive to accumulate, and assume that agents aim to maximise their wealth subject to reaching a minimum consumption standard.

Let $\mathcal{N}_t = \{1, \ldots, N_t\}$ be the set of agents at t with generic element ν. At the beginning of t, each $\nu \in \mathcal{N}_t$ is endowed with a (possibly zero) vector of produced goods, $\omega_{t-1}^{\nu} \in \mathbb{R}_+^n$ and one unit of labour.[11]

We follow Roemer [35, 37] in making the time structure of production explicit: "Time is essential in production, in the sense that capitalists must pay today for inputs before revenues are received tomorrow." (Roemer [35], p.506) Moreover, "there is no financial capital market: capitalists are limited in the extent of their production by the level of internal finance. [Because] allowing capitalists to borrow from each other ... does not change the results reported here." (ibid.) Within each production period t, market exchanges take place at two points in time: at the beginning of t, productive inputs are traded at prices $p_t^b \in \mathbb{R}_+^n$ and labour contracts are signed; at the end of t, outputs are traded at prices $p_t \in \mathbb{R}_+^n$ and workers are paid the nominal wage $w_t \geq 0$.

At the beginning of every t, each $\nu \in \mathcal{N}_t$ must form expectations $(p_t^{e\nu}, w_t^{e\nu})$ about (p_t, w_t). Because agents have the same preferences and possess the same information, we

[8] Vector inequalities: for all $x, y \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

[9] We follow the literature and focus on circulating capital but our analysis can be extended to models with fixed capital. See, for example, Roemer [34], Bidard [4], Flaschel et al [11], and Kiedrowski [21].

[10] This is a generalisation of Jones's [19] model of 'ideas'. We assume that innovations are discovered at the end of a given production period, and can only be used in the following period. The assumption that only one new technique can emerge at a time is for simplicity and yields no loss of generality.

[11] In every period t, we take the distribution of endowments as exogenously given and abstract from all issues related to bequests and the endowment of newly born agents. This is without any loss of generality and none of our results depends on it.

shall assume them to have identical expectations and drop the superscript ν for simplicity: $(p_t^{e\nu}, w_t^{e\nu}) = (p_t^e, w_t^e)$ for all $\nu \in \mathcal{N}_t$.

Given $\left(p_t^b, p_t^e, w_t^e\right)$, at the beginning of each t, every agent $\nu \in \mathcal{N}_t$ chooses her labour supply, l_t^ν, and uses wealth, $W_t^\nu = p_t^b \omega_{t-1}^\nu$, either to buy goods δ_t^ν (spending $p_t^b \delta_t^\nu$) for sale at the end of t or to finance production. In the latter case, each agent chooses a technique, $(A_t^\nu, L_t^\nu) \in \mathcal{P}_t$, which is activated at level x_t^ν by investing (part of) W_t^ν to finance operating costs, $p_t^b A_t^\nu x_t^\nu$, and by hiring workers $L_t^\nu x_t^\nu$, which are paid ex post the expected amount $w_t^e L_t^\nu x_t^\nu$.[12] Thus, expected gross revenue at the end of t is $p_t^e x_t^\nu + w_t^e l_t^\nu + p_t^e \delta_t^\nu$, which is used to pay wages and finance accumulation, $p_t^e \omega_t^\nu$, subject to purchasing a consumption bundle $b \in \mathbb{R}_+^n$, $b > \mathbf{0}$, per unit of labour performed.[13]

Let $\triangle \equiv \left\{p \in \mathbb{R}_+^n \mid pb = 1\right\}$. In every t, given $\left(p_t^b, p_t^e, w_t^e\right) \in \triangle^2 \times \mathbb{R}_+$, agents are assumed to choose (A_t^ν, L_t^ν), $\xi_t^\nu \equiv (x_t^\nu, l_t^\nu, \delta_t^\nu)$, and ω_t^ν to solve:[14]

$$MP_t^\nu : \max_{(A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu} p_t^e \omega_t^\nu$$

subject to

$$\begin{aligned} \left[p_t^e - w_t^e L_t^\nu\right] x_t^\nu + w_t^e l_t^\nu + p_t^e \delta_t^\nu &= p_t^e b l_t^\nu + p_t^e \omega_t^\nu && (1) \\ p_t^b A_t^\nu x_t^\nu + p_t^b \delta_t^\nu &= p_t^b \omega_{t-1}^\nu, && (2) \\ 0 &\leqq l_t^\nu \leqq 1, && (3) \\ (A_t^\nu, L_t^\nu) &\in \mathcal{P}_t; && (4) \\ x_t^\nu, \delta_t^\nu, \omega_t^\nu &\in \mathbb{R}_+^n. && (5) \end{aligned}$$

In other words, we focus on a temporary resource allocation problem whereby agents choose an optimal plan in each production period. The temporary equilibrium framework allows us to analyse the changes in optimal behaviour, and equilibrium allocation, sparked by unforeseen innovations.

Finally, let $v_t \equiv L_t(I - A_t)^{-1}$ denote the standard vector of employment multipliers. In the rest of the paper, we assume that for all $(A_t, L_t) \in \mathcal{B}_t$, $1 > v_t b$ holds: this is a basic condition for the productiveness of the economy.

3 Equilibrium

An economy at t is described by $\mathcal{N}_t$, $\mathcal{B}_t$, b, and $\Omega_{t-1} \equiv \left(\omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_t} \in \mathbb{R}_+^{nN_t}$ and is denoted as $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Let $x_t \equiv \sum_{\nu \in \mathcal{N}_t} x_t^\nu$, and let a similar notation hold for δ_t, ω_t, and l_t. Following Roemer [35, 37], the equilibrium notion of this economy can be defined.

[12] The model can be extended to allow agents to operate production activities with their own capital as self-employed producers. Given the convexity of the optimisation programme MP_t^ν below, this makes no difference to our results.

[13] Given our analytical focus on the relation between technical change and profitability, we do not explicitly analyse the agents' consumption choices and treat b as a parameter. This is without major loss of generality: allowing agents' optimal consumption bundle to adjust to price changes would hardly make a difference for our results, while significantly increasing technicalities. But all of the key insights of the paper concerning the effects of localised innovations – including the indeterminacy of equilibrium distribution, the possibility of technological unemployment, the non-existence of equilibrium, and the falling profit rate, – continue to hold, with appropriate adjustments, even if aggregate consumption is allowed to vary. For an analysis of consumption choices within a classical model with optimising agents, see Veneziani and Yoshihara [44], Galanis et al [14], and Yoshihara and Kwak [46].

[14] Constraints (1) and (2) are written as equalities without loss of generality.

Definition 1 *A competitive equilibrium (CE) for $E(\mathcal{N}_t;\mathcal{B}_t;b;\Omega_{t-1})$ is a vector $\left(p_t^b,p_t,w_t\right)\in\triangle^2\times\mathbb{R}_+$ and associated $((A_t^\nu,L_t^\nu);\xi_t^\nu;\omega_t^\nu)_{\nu\in\mathcal{N}_t}$ such that:*

(D1a) $((A_t^\nu,L_t^\nu);\xi_t^\nu;\omega_t^\nu)$ solves MP_t^ν, for all $\nu\in\mathcal{N}_t$ (individual optimality);

(D1b) $\sum_{\nu\in\mathcal{N}_t}A_t^\nu x_t^\nu+\delta_t\leqq\omega_{t-1}$ (social feasibility of production);

(D1c) $\sum_{\nu\in\mathcal{N}_t}L_t^\nu x_t^\nu=l_t$ (labour market);

(D1d) $x_t+\delta_t\geqq\sum_{\nu\in\mathcal{N}_t}bL_t^\nu x_t^\nu+\omega_t$ with $x_t>\mathbf{0}$ (commodity markets);

(D1e) $p_t=p_t^e=p_t^b$ and $w_t=w_t^e$ (realised expectations).

In other words, at a CE, (D1a) all agents optimise; (D1b) aggregate capital is sufficient for production plans; (D1c) the labour market clears; (D1d) the total supply of commodities is sufficient for consumption and accumulation plans; and (D1e) agents' expectations are realised ex post. For the sake of notational simplicity, because at a CE expectations are realised and $p_t=p_t^b=p_t^e$, we shall write the price vector as $(p_t,w_t)\in\triangle\times\mathbb{R}_+$.

The concept of CE is a temporary equilibrium notion which focuses on each period in isolation and it is thus conceptually related to equilibrium notions pioneered by Hicks [18], and later formalised by Hahn [17], Radner [32], Grandmont [16] and, in the context of activity analysis, Morishima [27], among others, who considered sequence of markets at successive dates, none of which is complete in the Arrow-Debreu sense. The dynamic evolution of the economy can thus be conceived of as a sequence of temporary equilibria. To assume that markets are not complete in the Arrow-Debreu sense captures the idea that technical progress – especially that of the Schumpeterian kind – is inherently unforeseeable and the appearance of innovations forces agents to revise their plans sequentially. Modelling the economy as a sequence of markets is therefore a natural choice given our focus on the general equilibrium effects of localised technical progress on distribution.[15]

Further, although classes are not the main focus of this paper, it is worth noting that if Definition 1 is adopted, then it is possible to extend the classic analysis developed by Roemer ([37], chapter 4) to derive the complete class structure of the economy that emerges in equilibrium in every t. In particular, based on Roemer's [37] definition of class, one can prove that at any given t, at a CE an agent ν belongs to the capitalist class if and only if $L_t^\nu x_t^\nu>l_t^\nu$, while she is part of the working class if and only if $L_t^\nu x_t^\nu<l_t^\nu$.[16] As shown by Veneziani [43], if a different equilibrium framework is adopted, the extension of Roemer's theory of class is not trivial, either formally or conceptually.

Two additional features of Definition 1 should be noted. First, it focuses on non-trivial allocations with a positive gross output vector, $x_t>\mathbf{0}$. This is without loss of generality because agents will optimally activate all sectors if the profit rate is positive; and even if the profit rate is zero, $x_t>\mathbf{0}$ can always be the product of optimal choices, consistent with such a CE. Second, following Roemer [35, 37], we assume that agents have *stationary expectations* and expect beginning-of-period commodity prices to rule at the end of the period, $p_t^e=p_t^b$. This is "a standard assumption of the temporary equilibrium literature" (Roemer [35], p. 529) which can also be interpreted as imposing not overly demanding conditions on agents' rationality and foresight in expectation formation, consistent with a large literature on bounded rationality and behavioural economics.

[15]For an analysis of intertemporal general equilibrium from a classical perspective, see Dana et al [9], Veneziani [43], Freni et. al. [13], Takahashi [42], Veneziani and Yoshihara [44], and Galanis et al [14].

[16]This result is a straightforward extension of Theorem 4.4 in Roemer [37].

For all $(p_t, w_t) \in \triangle \times \mathbb{R}_+$ and $(A, L) \in \mathcal{P}_t$, let $\pi_{it}^{(p_t,w_t)}(A, L) \equiv \frac{p_{it} - p_t A_i - w_t L_i}{p_t A_i}$, $i = 1, \ldots, n$; let $\pi_t^{(p_t,w_t)}(A, L) \equiv \max_{i=1,\ldots,n} \pi_{it}^{(p_t,w_t)}(A, L)$; and let $\pi_t^{\max} \equiv \max_{(A,L)\in\mathcal{P}_t} \pi_t^{(p_t,w_t)}(A, L)$.[17]

Lemma 1 derives some properties of the optimal solution to MP_t^ν.

Lemma 1 *Let $\left((p_t, w_t), ((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. For all $\nu \in \mathcal{N}_t$: (i) if $x_t^\nu \geq \mathbf{0}$ then $\pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) = \pi_t^{\max}$; and (ii) if $\pi_t^{\max} > 0$, then $p_t A_t^\nu x_t^\nu = p_t \omega_{t-1}^\nu$ and if $w_t > p_t b$, then $l_t^\nu = 1$.*

Proof: *Part (i).* Constraints (1)-(2) imply that at a CE the following equation holds:

$$p_t \omega_t^\nu = [p_t - p_t A_t^\nu - w_t L_t^\nu]\, x_t^\nu + (w_t - p_t b)\, l_t^\nu + p_t \omega_{t-1}^\nu, \forall \nu \in \mathcal{N}_t. \tag{6}$$

By equation (6), $x_t^\nu \geq \mathbf{0}$ implies $\pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) \geqq 0$. Suppose, by way of contradiction, that $\pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) \neq \pi_t^{\max}$. Then, there exists $(A', L') \in \mathcal{B}_t$ such that $\pi_t^{(p_t,w_t)}(A', L') > \pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) \geqq 0$. Let $x' \in \mathbb{R}^n$ be such that $x_i' > 0$ if $\pi_t^{(p_t,w_t)}(A', L') = \frac{p_{it} - p_t A_i' - w_t L_i'}{p_t A_i'}$ and $x_i' = 0$ otherwise; and $p_t A' x' = p_t \omega_{t-1}^\nu$. By the definition of $\pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu)$, it follows that $[p_t - p_t A_t^\nu - w_t L_t^\nu]\, x_t^\nu \leqq \pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) p_t A_t^\nu x_t^\nu$. By construction, $[p_t - p_t A' - w_t L']\, x' = \pi_t^{(p_t,w_t)}(A', L') p_t A' x' > \pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) p_t A_t^\nu x_t^\nu$. Hence, by equation (6), (A_t^ν, L_t^ν) cannot be part of an optimal solution.

Part (ii). It follows immediately from equation (6) noting that $[p_t - p_t A_t^\nu - w_t L_t^\nu]\, x_t^\nu > 0$ if and only if $\pi_t^{(p_t,w_t)}(A_t^\nu, L_t^\nu) > 0$. ∎

In principle, in equilibrium different production techniques may be used. By Lemma 1, however, and noting that at a CE $x_t > \mathbf{0}$, they all yield the same (maximum) profit rate. Hence, we shall henceforth assume without loss of generality that all agents who activate some production process opt for the same (A_t, L_t), and drop the superscript ν.

Lemma 2 proves that equilibrium prices are strictly positive and competition leads to the equalisation of sectoral profit rates in equilibrium.

Lemma 2 *Let $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then $\pi_t^{\max} \geqq 0$, $p_t = (1 + \pi_t^{\max})\, p_t A_t + w_t L_t$, $p_t > \mathbf{0}$, and $w_t \geqq p_t b$. If $w_t = p_t b$, then $\pi_t^{\max} > 0$.*

Proof: First, by equation (6) if $\pi_t^{\max} < 0$ then $x_t^\nu = \mathbf{0}$ for all ν, which contradicts $x_t > \mathbf{0}$. Second, since only sectors yielding the maximum profit rate are activated at the solution to MP_t^ν, $x_t > \mathbf{0}$ implies that $p_t = (1 + \pi_t^{\max})\, p_t A_t + w_t L_t$ holds. Third, by the productiveness and the indecomposability of A_t, $(I - A_t)$ is invertible with $(I - A_t)^{-1} > \mathbf{0}$ (Kurz and Salvadori [24]; Theorem A.3.5). Hence, $p_t = \pi_t^{\max} p_t A_t (I - A_t)^{-1} + w_t L_t (I - A_t)^{-1} > \mathbf{0}$. Fourth, $x_t > \mathbf{0}$ implies $L_t x_t > 0$. Hence by (D1c) and equation (6), it must be $w_t \geqq p_t b$.

Finally, suppose, by way of contradiction, that $w_t = p_t b$ but $\pi_t^{\max} \leqq 0$. Because $p_t = (1 + \pi_t^{\max})\, p_t A_t + w_t L_t$, $\pi_t^{\max} \leqq 0$ implies $p_t \leqq p_t A_t + w_t L_t$ or $p_t \leqq w_t L_t (I - A)^{-1} = w_t v_t$. Post-multiplying both sides of the latter inequality by $b > \mathbf{0}$, we obtain $p_t b \leqq w_t v_t b$, and the contradiction ensues from $v_t b < 1$. ∎

For each $(A, L) \in \mathcal{B}_t$, let

$$F(\pi; (A, L)) = \begin{cases} \frac{1}{L[I-(1+\pi)A]^{-1} b} & \text{if } \pi \in [0, \Pi(A)), \\ 0 & \text{if } \pi = \Pi(A). \end{cases}$$

[17] The maximum profit rate $\pi_t^{\max}$ is well defined as $\mathcal{B}_t$ is finite.

The *wage-profit curve (WPC) associated with* $(A, L) \in \mathcal{B}_t$ can be defined as follows:

$$\pi w(A, L) \equiv \left\{(\pi, w) \in \mathbb{R}_+^2 \mid w = F(\pi; (A, L)) \text{ for } \pi \in [0, \Pi(A)]\right\}.$$

For all $\pi \in [0, \Pi(A))$, $L[I - (1+\pi)A]^{-1} = L\left[\sum_{k=0}^{\infty} ((1+\pi)A)^k\right]$, and for any given $(A, L) \in \mathcal{B}_t$, the WPC is the graph of $F(\pi; (A, L))$ which is continuous and monotonically decreasing at every $\pi \in [0, \Pi(A))$ (Kurz and Salvadori [24]; Theorem A.3.3).

The *wage-profit frontier (WPF) associated with* $\mathcal{B}_t$ can be defined as follows:

$$\begin{aligned}\pi w(\mathcal{B}_t) \equiv & \left\{(\pi, w) \in \mathbb{R}_+^2 \mid \exists (A, L) \in \mathcal{B}_t : (\pi, w) \in \pi w(A, L) \right. \\ & \left. \& \ \forall (A', L') \in \mathcal{B}_t, \forall (\pi', w') \in \pi w(A', L') : w' = w \Rightarrow \pi' \leqq \pi\right\}.\end{aligned}$$

In other words, the WPF is the envelope of the various WPCs and therefore it also identifies a continuous, inverse relation between w and π.

The concepts of the WPC and WPF provide the analytical tools to examine the optimal choice of technique and the interaction between technical progress and distribution. For in equilibrium only techniques that lie on $\pi w(\mathcal{B}_t)$ will be adopted. Formally:

Lemma 3 *A technique* $(A, L) \in \mathcal{B}_t$ *with* $p = (1+\pi)pA + wL$ *for some* $(p, w) \in \triangle \times \mathbb{R}_+, w > 0$ *is such that* $p \leqq (1+\pi)pA' + wL'$ *for all* $(A', L') \in \mathcal{B}_t$ *if and only if* $(\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_t)$.

Proof: See Kurz and Salvadori ([24]; Theorem 5.1(b)). ■

In other words, a technique (A, L) is cost minimising, and is therefore adopted, if no other technique in $\mathcal{B}_t$ allows for a wage rate higher than w at π.

For each (A, L), the intercepts of $\pi w(A, L)$ on the vertical axis (w) and on the horizontal axis (π) are, respectively, the points $\left(0, \frac{1}{vb}\right)$ and $(\Pi(A), 0)$. Therefore, for any $(A, L), (A', L') \in \mathcal{B}_t$, if $v > v'$ and $A \leq A'$, then $\left(0, \frac{1}{vb}\right) \leq \left(0, \frac{1}{v'b}\right)$ and $(\Pi(A), 0) \geq (\Pi(A'), 0)$, and $\pi w(A, L)$ and $\pi w(A', L')$ intersect at least once, and quite possibly more than once. Finally, given $\pi w(A, L)$ and given $(\pi, w) \in \mathbb{R}_+^2$, let

$$\pi w(A, L; (\pi, w)) \equiv \{(\pi', w') \in \pi w(A, L) \mid (\pi', w') \geq (\pi, w)\}.$$

The WPF is conceptually equivalent to the factor price frontier used in standard microeconomics, but there are some key differences. For example, well-known paradoxes in capital theory, such as reswitching and capital reversing can only be analysed focusing on the WPF (Sraffa [39], Kurz and Salvadori [24]). Perhaps more importantly for our analysis, in the standard approach technical progress is conceived of as yielding an outward shift of the *whole* factor prices frontier (and possibly a change in its shape). Thus, for any (π, w) in the original frontier, there is (π', w') in the new frontier such that $(\pi', w') \geq (\pi, w)$. In contrast, as shown below, a localised innovation $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ will typically have an analogous effect only in a *neighbourhood* of the original equilibrium distribution $(\pi, w) \in \pi w(\mathcal{B}_{t-1})$: there will be a *neighbourhood* of (π, w) in $\pi w(\mathcal{B}_{t-1})$ such that for any pair (π', w') in such neighbourhood, there exists $(\pi^*, w^*) \in \pi w(A^*, L^*; (\pi', w'))$.

We conclude this section deriving some key properties of competitive equilibria that will be useful in what follows.

Theorem 1 *Let* $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ *be a CE for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.
(i) If $\pi_t^{\max} > 0$ *and* $w_t > p_t b$*, then* $A_t x_t = \omega_{t-1}$ *and* $N_t = L_t x_t$.
(ii) If $A_t x_t = \omega_{t-1}$ *and* $N_t > L_t x_t$*, then* $w_t = p_t b$*. In contrast, if* $w_t = p_t b$*, then* $A_t x_t = \omega_{t-1}$.
(iii) If $A_t x_t \leq \omega_{t-1}$ *and* $N_t = L_t x_t$*, then* $\pi_t^{\max} = 0$*. In contrast, if* $\pi_t^{\max} = 0$*, then* $N_t = L_t x_t$.

Proof: *Part (i).* By Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$ and $l_t^\nu = 1$ for all $\nu \in \mathcal{N}_t$. Then, $p_t A_t x_t = p_t \omega_{t-1}$ holds, and by (D1c), $L_t x_t = l_t = N_t$. By Lemma 2, $p_t > \mathbf{0}$. Therefore, $p_t A_t x_t = p_t \omega_{t-1}$ and (D1b) imply $A_t x_t = \omega_{t-1}$.
Part (ii). First, let $A_t x_t = \omega_{t-1}$ and $N_t > L_t x_t$, and suppose, contrary to the statement, that $w_t > p_t b$. Then, by Lemma 1, $l_t^\nu = 1$, all $\nu \in \mathcal{N}_t$, which contradicts (D1c). Next, let $w_t = p_t b$. By Lemma 2, $\pi_t^{\max} > 0$, which further implies that $p_t A_t x_t = p_t \omega_{t-1}$ by Lemma 1. Moreover, by Lemma 2, $p_t > \mathbf{0}$. Therefore, by (D1b), $A_t x_t = \omega_{t-1}$.
Part (iii). First, let $A_t x_t \leq \omega_{t-1}$ and $N_t = L_t x_t$, and suppose, contrary to the statement, that $\pi_t^{\max} > 0$. Then, by Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$, for all $\nu \in \mathcal{N}_t$, and so $p_t A_t x_t = p_t \omega_{t-1}$. Therefore since by Lemma 2 $p_t > \mathbf{0}$ holds, (D1b) implies $A_t x_t = \omega_{t-1}$, a contradiction. Next, let $\pi_t^{\max} = 0$. It implies $w_t > p_t b$ by Lemma 2. Then, by Lemma 1, $l_t^\nu = 1$, all $\nu \in \mathcal{N}_t$, which implies $N_t = l_t = L_t x_t$ by (D1c). ∎

4 Profitable innovations and the Okishio Theorem

4.1 The status quo at the point in time when innovations emerge

Innovations do not occur in a vacuum. When a new production technique emerges, an economy is already set on a given growth path, and firms decide whether to adopt it by comparing it with the technique(s) currently in use. Innovations without the potential to increase profits relative to the status quo are unlikely to be adopted, and will not move the economy away from its growth trajectory. In this subsection, we rigorously formalise the notion of the status quo, that is, the equilibrium path of the economy at the point in time when the innovation emerges.

In order to abstract from other factors – such as those related to the dynamics of productive inputs – we consider a subset of equilibria such that, *absent any innovation*, equilibrium prices would be invariant across two periods. Formally:

Definition 2 *Let* $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi_{t-1}^\nu; \omega_{t-1}^\nu)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a CE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *The CE is persistent, or a PCE, if and only if there is a profile* $(\xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ *such that* $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ *is a CE for* $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$ *with* $\omega_{t-1} = \sum_{\nu \in \mathcal{N}_{t-1}} \omega_{t-1}^\nu$.

If the economy is at a PCE at $t-1$, then neither equilibrium prices (p_{t-1}, w_{t-1}) nor the production technique (A_{t-1}, L_{t-1}) need to vary at t, as long as no innovation emerges between the two periods. Therefore, the notion of PCE is primarily an analytical device to examine the effect of technical progress in vitro. It describes *the status quo* and starting point for the analysis of the effects of innovations: a possibly counterfactual allocation that would emerge at t if the economy had the same production set as at $t-1$, $\mathcal{B}_t = \mathcal{B}_{t-1}$.

Absent technical progress, the conditions for the persistence of a CE are not particularly strong, as they basically require capital accumulation to appropriately adjust to changes in demographic conditions. Formally:[18]

Proposition 1 *Let* $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ *be a CE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$*, and let* $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; (\omega_{t-1}^{\nu})_{\nu\in\mathcal{N}_{t-1}})$ *be the economy in period* t*. Then,*
(i) if $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} > p_{t-1}b$*, then this CE is persistent if and only if* $A_{t-1}^{-1}\omega_{t-1} > \mathbf{0}$ *and* $N_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1}$*;*
(ii) if $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} = p_{t-1}b$*, then this CE is persistent if and only if* $A_{t-1}^{-1}\omega_{t-1} > \mathbf{0}$ *and* $N_t \geqq L_{t-1}A_{t-1}^{-1}\omega_{t-1}$*;*
(iii) if $\pi_{t-1}^{\max} = 0$ *and* $w_{t-1} > p_{t-1}b$*, then this CE is persistent if and only if there exists* $\delta \geqq \mathbf{0}$ *such that* $A_{t-1}^{-1}(\omega_{t-1} - \delta) > \mathbf{0}$ *and* $N_t = L_{t-1}A_{t-1}^{-1}(\omega_{t-1} - \delta)$*.*

The concept of PCE plays a key analytical role as a benchmark in our analysis: *in all of our definitions and results*, we assume that *the economy starts off from a PCE at* $t-1$. Then, we define the properties of innovations emerging at t relative to the *status quo technique* – the technique adopted at $t-1$, – and analyse the effect of innovations on equilibrium wages and profits relative to the *status quo* at $t-1$, – a status quo which could persist, by construction, absent any innovations.

The existence of a PCE at $t-1$ is therefore a premise for all of our formal results concerning the effect of innovations at t. In the Appendix, we prove that this premise is not empty, and derive a full characterisation of the conditions guaranteeing the existence of all types of PCEs considered in our results. Hence, in the remainder of the paper, for the sake of brevity, and notational clarity, we shall write "*Suppose the economy is at a PCE at* $t-1$" to mean "*Let* $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^{\nu};\ \omega_{t-1}^{\nu}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ *be a PCE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$."

4.2 Profitable innovations

The concept of PCE allows us to analyse what may be thought of as Schumpeterian innovations: new techniques that create unforeseen profit opportunities, disrupt existing production processes, and cause fundamental shifts in the key distributive variables.

To see this, suppose that the economy is at a PCE in period $t-1$ and technical progress occurs before productive inputs are bought and production starts in period t. Not all innovations $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ alter incentives and lead agents to deviate from the PCE. If $\pi_t^{(p_{t-1}, w_{t-1})}(A^*, L^*) \leqq \pi_{t-1}^{\max} = \pi_{t-1}^{(p_{t-1}, w_{t-1})}(A_{t-1}, L_{t-1})$ holds, then (p_{t-1}, w_{t-1}) and (A_{t-1}, L_{t-1}) would still constitute a CE in period t. This motivates our focus on innovations that are profitable in the following sense:[19]

Definition 3 *Suppose the economy is at a PCE at* $t-1$*. An innovation* $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ *is profitable relative to the status quo technique (henceforth, profitable) if and only if:*

$$\left(1 + \pi_{t-1}^{\max}\right) p_{t-1}A^* + w_{t-1}L^* \leq \left(1 + \pi_{t-1}^{\max}\right) p_{t-1}A_{t-1} + w_{t-1}L_{t-1}.$$

[18] Proposition 1 follows immediately from Theorem 1 and its proof is therefore omitted.

[19] Definition 3 generalises the notion of profitable, or *viable*, technical change that is standard in the literature. See, for example, Okishio [30], Roemer [33, 34, 36], and Flaschel et al [11].

In other words, a new technique is profitable if a producer can expect *extra profits* by unilaterally switching to it from the status quo. Given that the economy starts off at a PCE, if the new technique (A^*, L^*) emerging at t did not reduce costs at prices (p_{t-1}, w_{t-1}) in at least one sector, then the old technique (A_{t-1}, L_{t-1}) would still be optimal and the equilibrium price vector could persist. Thus, Definition 3 characterises a sufficient condition for the PCE to disappear. "A [profitable] innovation will immediately be adopted by capitalists who treat prices as given, as they will make super-profits from its operation" (Roemer [36], p.452). Indeed, as shown by Lemma 4 below, if technical progress affects only one production process, then the inequality in Definition 3 is also necessary for the PCE to disappear.

Two points are worth noting about Definition 3. First, the premise that the economy is at a PCE in period $t-1$ is crucial. If the CE were not persistent, then Definition 3 would not capture a relevant condition for innovations to disrupt behaviour, as the economy may move to a different equilibrium because of demographic factors and/or due to capital accumulation. Similarly, the fact that the new technique would have been profitable at last period's prices would be immaterial for today's decisions.

Second, Definition 3 does not tell us anything, a priori, about the effect of technical progress on wages and profits. For, on the one hand, the condition in Definition 3 is not sufficient to guarantee that the new technique will be adopted at the new, generically different, equilibrium prices (p_t, w_t) in period t. On the other hand, even if the new technique (A^*, L^*) is indeed optimal at (p_t, w_t), $\pi_t^{\max} = \pi_t^{(p_t,w_t)}(A^*, L^*)$ may be higher or lower than $\pi_{t-1}^{\max} = \pi_t^{(p_{t-1},w_{t-1})}(A_{t-1}, L_{t-1})$. Therefore it is unclear whether technical progress has a positive effect on profitability, as Schumpeter suggested, or rather it may drive the equilibrium profit rate to fall, as Marx argued.

Lemma 4 forcefully illustrates both points, and the relevance of Definition 3, as it shows that profitable innovations shift the WPF out in a neighbourhood of the old equilibrium.

Lemma 4 *Suppose the economy is at a PCE in period $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$. If (A^*, L^*) is profitable, then $\varnothing \neq \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subseteq \pi w\left(\mathcal{B}_t\right) \setminus \pi w\left(\mathcal{B}_{t-1}\right)$. Conversely, if $(A_j^*, L_j^*) \neq (A_{jt-1}, L_{jt-1})$ for some j and $(A_i^*, L_i^*) = (A_{it-1}, L_{it-1})$ for all $i \neq j$, and $\varnothing \neq \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subseteq \pi w\left(\mathcal{B}_t\right) \setminus \pi w\left(\mathcal{B}_{t-1}\right)$, then (A^*, L^*) is profitable.*

Proof: Let $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. By Lemma 3, $\left(\pi_{t-1}^{\max}, w_{t-1}\right) \in \pi w\left(A_{t-1}, L_{t-1}\right) \cap \pi w\left(\mathcal{B}_{t-1}\right)$.

1. Let (A^*, L^*) be profitable. Then, it follows from $\left(1+\pi_{t-1}^{\max}\right) p_{t-1} A^* + w_{t-1} L^* \leq \left(1+\pi_{t-1}^{\max}\right) p_{t-1} A_{t-1} + w_{t-1} L_{t-1} = p_{t-1}$ and $w_{t-1} \geqq 1$ (by Lemma 2) that $p_{t-1}\left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right] > \mathbf{0}$, which in turn implies that $\left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]$ is invertible with $\left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]^{-1} > \mathbf{0}$ by the indecomposability of A^* (Kurz and Salvadori [24]; Theorem A.3.5). Then, profitability implies $p_{t-1} \geq w_{t-1} L^* \left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]^{-1}$, which in turn implies that

$$1 = p_{t-1} b > w_{t-1} L^* \left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]^{-1} b = \frac{w_{t-1}}{\widehat{w}} \text{ for } \widehat{w} \equiv \frac{1}{L^* \left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]^{-1} b}.$$

Then, $\widehat{w} > w_{t-1}$. Moreover, let $\widehat{p} \equiv \widehat{w} L^* \left[I - \left(1+\pi_{t-1}^{\max}\right) A^*\right]^{-1} > \mathbf{0}$. Then, $\widehat{p} \in \Delta$.

Clearly, $\left(\pi_{t-1}^{\max}, \widehat{w}\right) \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$, and also $\left(\pi_{t-1}^{\max}, \widehat{w}\right) \in \pi w\left(\mathcal{B}_t\right) \setminus \pi w\left(\mathcal{B}_{t-1}\right)$ holds as $\left(\pi_{t-1}^{\max}, w_{t-1}\right) \in \pi w\left(\mathcal{B}_{t-1}\right)$. Moreover, for any $(\pi', w') \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$,

$(\pi', w') \notin \pi w(\mathcal{B}_{t-1})$ holds, because $(\pi', w') \geq \left(\pi_{t-1}^{\max}, w_{t-1}\right)$ and $F(\pi; (A', L'))$ is strictly decreasing and continuous for each $(A', L') \in \mathcal{B}_{t-1}$. Therefore, $(\pi', w') \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$.

2. Let (A^*, L^*) be such that $(A_j^*, L_j^*) \neq (A_{jt-1}, L_{jt-1})$ for some j and $(A_i^*, L_i^*) = (A_{it-1}, L_{it-1})$ for all $i \neq j$, and let $\varnothing \neq \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subseteq \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$. Assume, ad absurdum, that (A^*, L^*) is not profitable. Then, $p_{t-1}\left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right] \leqq w_{t-1} L^*$ holds. Again, as A^* is indecomposable and $\varnothing \neq \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right) \subseteq \pi w(\mathcal{B}_t)$, $\left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1}$ exists and is strictly positive. Post-multiplying the latter inequality by $\left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1} b > \mathbf{0}$, we obtain $1 \leqq w_{t-1} L^* \left[I - \left(1 + \pi_{t-1}^{\max}\right) A^*\right]^{-1} b \equiv \frac{w_{t-1}}{\widehat{w}}$. Yet, as $\left(\pi_{t-1}^{\max}, \widehat{w}\right) \in \pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$, $\widehat{w} \leqq w_{t-1}$ implies a contradiction. ■

The effect of profitable innovations on the WPF in a neighbourhood of a status quo with $\pi_{t-1}^{max} > 0$ and $w_{t-1} > 1$ is depicted in Figure 1.

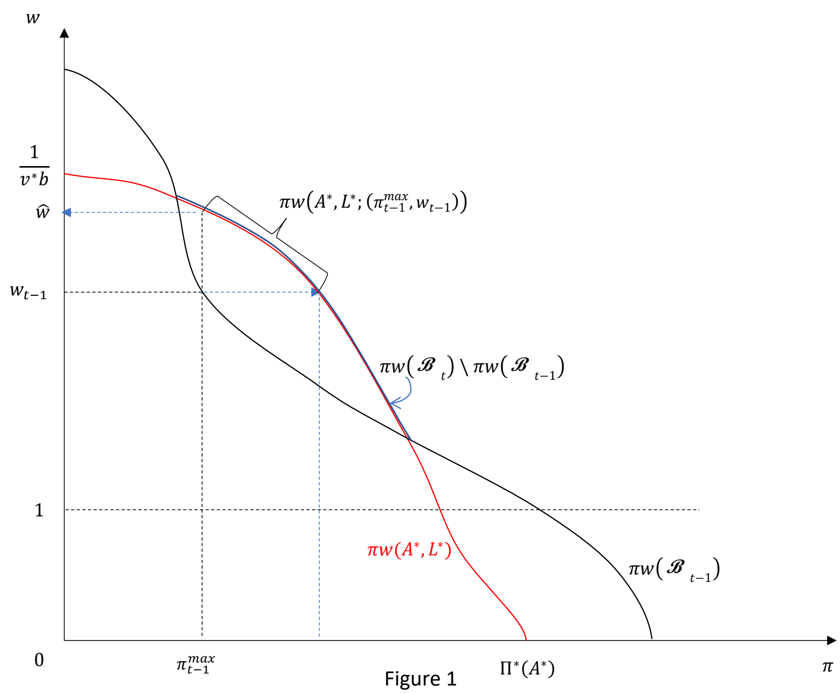

Figure 1: The effect on the WPF of an innovation $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ that is profitable relative to $(A_{t-1}, L_{t-1}) \in \mathcal{B}_{t-1}$ at (p_{t-1}, w_{t-1}) such that $\pi_{t-1}^{max} > 0$ and $w_{t-1} > 1$.

In a seminal contribution, Okishio [30] proved that if the wage rate is fixed at the subsistence level, then a profitable innovation always leads the equilibrium profit rate to increase, thus casting doubts on Marx's law of the falling rate of profit. Given Lemma 4, Theorem 2 generalises the Okishio Theorem (OT).[20]

Theorem 2 *Suppose the economy is at a PCE at $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ be profitable. If $\left((p_t, w_t), \left((A^*, L^*); \xi_t^\nu; \omega_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t \leqq w_{t-1}$, then $\pi_t^{\max} > \pi_{t-1}^{\max}$.*

[20]Theorem 2 is more general than standard versions of OT in that we adopt a general equilibrium concept which implies, but does not reduce to, the equalisation of sectoral profit rates.

Proof: If $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ then by Lemma 3, $(w_t, \pi_t^{\max}) \in \pi w(A^*, L^*) \cap \pi w(\mathcal{B}_t)$. If $w_t = w_{t-1}$, then the result follows immediately from Lemma 4. If $w_t < w_{t-1}$, the result follows noting that $F(\pi; (A^*, L^*))$ is continuous and monotonically decreasing with $F : [0, \Pi(A^*)] \to \left[0, \frac{1}{v^* b}\right]$ where $v^* = L^* [I - A^*]^{-1}$. ■

OT says that "viable [profitable] technical changes cause the wage-profit rate frontier to move outward, and therefore raise the equilibrium profit rate at constant real wages" (Roemer [36], p.451). Theorem 2 generalises this insight, which is far from obvious. As Roemer ([33], p.409) put it: "Clearly if a capitalist introduces a cost-reducing technical change his short-run profit rate rises. This, however, produces a disequilibrium; what the theorem says is that after prices have readjusted to equilibrate the profit rate again, the new profit rate will be higher than the old rate."

Nonetheless, Theorem 2 holds under rather restrictive assumptions. It proves that *if* (i) a new equilibrium exists in which (ii) the new technique is adopted and (iii) the wage rate does not increase, then OT holds. As shown below, the implications of localised technical change are much less clear-cut once conditions (i)-(iii) are relaxed.

Consider a simple one-good economy with two inputs, capital and labour. In the standard approach, technical progress implies an inward shift of the isoquants, technical change is always cost-reducing, and innovations are adopted in equilibrium. Assuming differentiability, the slope of the new isoquant at the point corresponding to the economy's capital and labour endowments represents the new, unique, equilibrium factor prices associated with increased production and the full employment of both inputs.

In contrast, a localised innovation $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ is a single new activity placed outside of the existing input-requirement set. The new input-requirement set is the convex hull of these two components, and therefore has a kink corresponding to the new technique, even if the original input-requirement set had a smooth boundary. As a result, the interaction of localised innovations, labour market conditions and maximising behaviour portrays a more complex, and arguably more realistic picture of technical progress. First, even if the new technique is profitable, it is not necessarily adopted in equilibrium. Further, it is not necessarily compatible with the full employment of capital and labour, which in turn makes the equilibrium transition more complicated. Finally, even if the universal adoption of the new technique was compatible with the full employment of capital and labour, it is not obvious what the new equilibrium price vector would be.

5 Technical progress and general equilibrium

In order to examine the relation between cost-reducing technical change and productivity in an activity analysis model, we follow Roemer [33] and define various properties of localised innovations $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ *relative to the status quo technique* $(A_{t-1}, L_{t-1}) \in \mathcal{B}_{t-1}$ (see also Flaschel et al. [11]).

Definition 4 *Suppose the economy is at a PCE in period $t-1$. Relative to the status quo technique (A_{t-1}, L_{t-1}), an innovation $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ is:*

(i) capital-using labour-saving (CU-LS) if and only if $A^* \geq A_{t-1}$ and $L^* \leq L_{t-1}$, with $A_i^* \geq A_{it-1}$ and $L_i^* < L_{it-1}$ for some i; and capital-saving labour-using (CS-LU) if and only if $A^* \leq A_{t-1}$ and $L^* \geq L_{t-1}$, with $A_i^* \leq A_{it-1}$ and $L_i^* > L_{it-1}$ for some i;

(ii) progressive if and only if $v^* < v_{t-1}$*; neutral if and only if* $v^* = v_{t-1}$*; and regressive if and only if* $v^* > v_{t-1}$*, where* $v_{t-1} \equiv L_{t-1}(I - A_{t-1})^{-1}$ *and* $v^* \equiv L^*(I - A^*)^{-1}$*.*

Two features of Definition 4 are worth stressing. First, in part (i) innovations are defined in physical, rather than monetary terms in order to abstract from the general equilibrium effects of technical change on prices. Only technical changes that are weakly monotonic in all produced inputs are considered. Although this may seem restrictive in an n-good space, it is in line with the definitions used in the literature, and with intuitive notions of the mechanisation process that has characterised much of capitalist development. Second, part (ii) provides a link between innovations and productivity: the adoption of a new technique is progressive if it leads to a uniform *decrease* in employment multipliers, and therefore to an *increase* in labour productivity. As Flaschel et al [11] show, these innovations expand the economy's production possibility frontier. Regressive technical changes have the opposite effect.[21]

In order to derive the next results, we impose more structure on technical progress and focus on technical changes whose main effect is on labour, rather than on capital inputs.

Definition 5 *Suppose the economy is at a PCE in period* $t-1$*. Let* $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ *be such that* $A_{it-1} \neq A_i^*$ *for some sector* i*. Then, the innovation is* labour inelastic *relative to the status quo technique* (A_{t-1}, L_{t-1}) *if and only if* $|L_{it-1} - L_i^*| > \left|L_{t-1}A_{t-1}^{-1}\left(A_{it-1} - A_i^*\right)\right|$ *for each* i *with* $A_{it-1} \neq A_i^*$*.*

The intuition is straightforward in a one-good economy: the innovation is labour inelastic relative to (A_{t-1}, L_{t-1}) if the percentage change in produced input is smaller than the percentage change in labour input. In an n-good economy, $(A_{it-1} - A_i^*)$ is the change in the *vector* of commodity inputs necessary to produce one unit of good i. The linear operator $L_{t-1}A_{t-1}^{-1}$ transforms the units of physical goods into labour: $L_{t-1}A_{t-1}^{-1}\left(A_{it-1} - A_i^*\right)$ represents the amount of labour necessary for the operation of the variational commodity inputs. Thus, Definition 5 states that the innovation is labour inelastic if and only if the change in the profile of commodity inputs measured in labour units is smaller than the change of direct labour input necessary to produce one unit of good i.

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with aggregate capital stocks $\omega_{t-1} \geq \mathbf{0}$, we define the set of activities (A, L) such that, given ω_{t-1}, all agents can reach subsistence b:

$$\mathcal{B}_t\left(\omega_{t-1}; b\right) \equiv \left\{(A, L) \in \mathcal{B}_t \mid A^{-1}\omega_{t-1} > \mathbf{0} \text{ and } \left(I - bL\right)A^{-1}\omega_{t-1} \geqq \mathbf{0}\right\}.$$

In other words, if $(A, L) \in \mathcal{B}_t\left(\omega_{t-1}; b\right)$ is adopted, then there exists a profile of actions $(x_t^\nu;\ l_t^\nu; \delta_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ satisfying (D1b) and (D1d).

5.1 Indeterminacy

In the analysis of the interaction between technical progress and the equilibrium distribution, it is natural to start from innovations which allow for the full employment of all factors of production. Theorem 3 shows that even in this special case, the distributive effects of technical progress are difficult to predict and may not be Pareto-improving.

[21] Part (ii) focuses on innovations that modify all employment multipliers in the same direction. As Roemer ([33], p.410) notes, this is without loss of generality if one considers innovations of the type described in part (i).

Theorem 3 *Suppose the economy is at a PCE in period $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ be profitable with $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$ and $N_t = L^* A^{*-1} \omega_{t-1}$. Then, $\pi w \left(A^*, L^*; \left(\pi^{\max}_{t-1}, w_{t-1}\right)\right) \neq \varnothing$ and for any $(\pi', w') \in \pi w \left(A^*, L^*; \left(\pi^{\max}_{t-1}, w_{t-1}\right)\right)$, there exists a CE $\left((p_t, w_t), \left((A^*, L^*); \xi^\nu_t; \omega^\nu_t\right)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $w_t = w'$ and $\pi^{\max}_t = \pi'$. Furthermore, if $\pi^{\max}_{t-1} > 0$ and $w_{t-1} > p_{t-1} b$, then there exist CEs with either $\pi^{\max}_t < \pi^{\max}_{t-1}$ or $w_t < w_{t-1}$.*

Proof: 1. Let $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi^\nu_{t-1}; \omega^\nu_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. By Lemma 3, $\left(\pi^{\max}_{t-1}, w_{t-1}\right) \in \pi w (A_{t-1}, L_{t-1}) \cap \pi w (\mathcal{B}_{t-1})$.

Because (A^*, L^*) is profitable, by Lemma 4, there exists $\widehat{w} > w_{t-1}$ such that for all $w \in [w_{t-1}, \widehat{w}]$, there exists $(\pi, w) \in \pi w (\mathcal{B}_t) \setminus \pi w (\mathcal{B}_{t-1})$ with $(\pi, w) \geq \left(\pi^{\max}_{t-1}, w_{t-1}\right)$.

2. Consider any $(\pi', w') \in \pi w (A^*, L^*) \cap \pi w (\mathcal{B}_t)$ such that $(\pi', w') \geq \left(\pi^{\max}_{t-1}, w_{t-1}\right)$. By Lemma 3, there is a $p' \in \triangle$ such that $p' = w' L^* \left[I - (1 + \pi') A^*\right]^{-1} > \mathbf{0}$ and (A^*, L^*) is optimal at (p', w'). Then, noting that $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, for all $\nu \in \mathcal{N}_t$, let

$$
\begin{aligned}
x^\nu_t &= \frac{p' \omega^\nu_{t-1}}{p' \omega_{t-1}} A^{*-1} \omega_{t-1} \geqq \mathbf{0}, \\
l^\nu_t &= 1, \\
\delta^\nu_t &= \mathbf{0}, \\
\omega^\nu_t &= \frac{p' x^\nu_t - w' L^* x^\nu_t + w' - 1}{(p' - L^*) A^{*-1} \omega_{t-1}} (I - bL^*) A^{*-1} \omega_{t-1} \geqq \mathbf{0}.
\end{aligned}
$$

Constraints (2)-(5) in MP^ν_t are clearly satisfied. Plugging $x^\nu_t, l^\nu_t = 1, \delta^\nu_t, \omega^\nu_t$ into constraint (1), and noting that $p' \in \triangle$, it is immediate to verify that the latter is also satisfied. Therefore $\left((A^*, L^*); \xi^\nu_t;\ \omega^\nu_t\right)$ solves MP^ν_t for all $\nu \in \mathcal{N}_t$, and (D1a) is satisfied.

Summing $x^\nu_t, l^\nu_t, \delta^\nu$ over ν, and noting that $N_t = L^* A^{*-1} \omega_{t-1}$, it follows that (D1b) and (D1c) are satisfied. To see that (D1d) is also satisfied observe that $\sum_{\nu \in \mathcal{N}_t} (w' L^* x^\nu_t - w') = 0$ because (D1b) holds, while $N_t = L^* A^{*-1} \omega_{t-1}$ and $x_t = A^{*-1} \omega_{t-1}$ imply $p' x_t - N_t = (p' - L^*) x_t$. Then $\sum_{\nu \in \mathcal{N}_t} \omega^\nu_t = (I - bL^*) A^{*-1} \omega_{t-1}$.

We conclude that $\left((p_t, w_t), \left((A^*, L^*); \xi^\nu_t;\ \omega^\nu_t\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $p_t = p'$, $w_t = w'$, and $\pi^{\max}_t = \pi'$.

3. If $\pi^{\max}_{t-1} > 0$ and $w_{t-1} > p_{t-1} b$, then by the continuity of $F(\pi; (A', L'))$ for all $(A', L') \in \mathcal{B}_t$, a straightforward modification of the argument in Lemma 4 can be used to show that there exists $(\pi^*, w^*) \in \pi w (A^*, L^*) \cap \pi w (\mathcal{B}_t)$ such that *either* $0 < \pi^{\max}_{t-1} - \pi^* < \varepsilon$ with $\pi^* \geqq 0$, *or* $0 < w_{t-1} - w^* < \varepsilon$ with $w^* \geqq 1$, for some $\varepsilon > 0$. The existence of a CE can then be proved as in step 2. ■

Theorem 3 suggests that when a profitable innovation guarantees the full employment of labour and capital, a new equilibrium emerges at t in which the new technique is indeed adopted. The effect of innovation on distribution is not clear a priori, however, because of the (infinitely) many profit rates and wage rates that can be supported in equilibrium. Interestingly, technical progress may even make some agents *strictly worse off* relative to the status quo, depending on their main source of income, as there exist equilibria at t with either $\pi^{\max}_t < \pi^{\max}_{t-1}$ or $w_t < w_{t-1}$.[22] The distributional outcome will depend on the

[22] To be precise, in period t, at a CE described in Theorem 3, an agent's income is $(1 + \pi^{\max}_t) p_t \omega^\nu_{t-1} + w_t$. Therefore, an innovation that disrupts the status quo makes agent ν worse off if and only if $\left[(1 + \pi^{\max}_{t-1}) p_{t-1} - (1 + \pi^{\max}_t) p_t\right] \cdot \omega^\nu_{t-1} > w_t - w_{t-1}$. Thus, agents whose income derives mostly from capital (labour) are worse off if the profit rate (wage rate) falls relative to the status quo.

actual equilibrium selection mechanism.[23]

It is worth emphasising, again, that this indeterminacy is due to the localised nature of innovations. It would arise even if the original set $\mathcal{B}_{t-1}$ contained a (possibly uncountably) infinite number of techniques, and we allowed consumption demand to vary with prices.

5.2 Technological unemployment

The previous subsection analyses equilibria in which a new technique is adopted and both capital and labour are fully employed. Yet this is by no means guaranteed in the case of localised technical progress, which may lead to technological unemployment – as conjectured by Acemoglu [2]. For example, even if the economy was originally at a CE with full employment, a CU-LS innovation may make labour relatively abundant *if the new technique is adopted*. As we have already noted, however, unlike in the standard analysis of technical progress, this is not necessarily the case: while the innovation is profitable at the status quo equilibrium prices, the very introduction of the new technique is likely to cause disequilibrium, which in turn would cause prices to change. Even though the WPC associated with the new, profitable technique will be part of the WPF in a neighbourhood of the original equilibrium, this is not necessarily true sufficiently far away from it.

Theorem 4 derives the conditions under which a profitable, CU-LS innovation is adopted at a new CE with labour unemployment.

Theorem 4 *Suppose the economy is at a PCE with $\pi_{t-1}^{\max} > 0$ and a sufficiently small $w_{t-1} - p_{t-1}b \geqq 0$ at $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be profitable, CU-LS, and labour inelastic with $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. Then there exists a CE $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ with $L^* x_t < N_t$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Moreover, for any CE with $L^* x_t < N_t$ in which (A^*, L^*) is adopted for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, $w_t = p_t b$ and $\pi_t^{\max} > \pi_{t-1}^{\max}$ hold.*

Proof: 1. Let $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi_{t-1}^\nu; \omega_{t-1}^\nu)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE with $\pi_{t-1}^{\max} > 0$ and a sufficiently small $w_{t-1} - p_{t-1}b \geqq 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. By Proposition 1, $N_t \geqq L_{t-1} A_{t-1}^{-1} \omega_{t-1}$ and there exist $\left(\hat{\xi}_t^\nu, \hat{\omega}_t^\nu\right)_{\nu \in \mathcal{N}_t}$ such that $\hat{x}_t > \mathbf{0}$ with $A_{t-1}\hat{x}_t = \omega_{t-1}$, and $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \hat{\xi}_t^\nu; \hat{\omega}_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.

2. We prove that $N_t > L^* A^{*-1} \omega_{t-1}$ holds. Because (A^*, L^*) is CU-LS, $L^* \hat{x}_t < L_{t-1} \hat{x}_t \leqq N_t$. Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, let $x_t \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$. If $x_t \leqq \hat{x}_t$, then the result follows immediately. Therefore, suppose $x_t \nleqq \hat{x}_t$.

Suppose, by way of contradiction, that $L^* x_t \geqq N_t$. Given $L^* \leq L_{t-1}$ and $x_t > \mathbf{0}$, this implies $L_{t-1} x_t > L^* x_t \geqq N_t$. Next, $A^* x_t = \omega_{t-1}$ and $A_{t-1}\hat{x}_t = \omega_{t-1}$ imply $N_t \geqq L_{t-1} A_{t-1}^{-1} A_{t-1} \hat{x}_t = L_{t-1} A_{t-1}^{-1} A^* x_t$. Hence, $L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t < 0$. Because the innovation is labour inelastic, $(L^* - L_{t-1}) x_t < L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t$, which implies $L^* x_t < L_{t-1} A_{t-1}^{-1} A^* x_t \leqq N_t$ which yields the desired contradiction.

3. Because (A^*, L^*) is profitable, Lemma 4 implies that there exists $\widehat{w} > w_{t-1}$ such that for all $w \in [w_{t-1}, \widehat{w}]$, there exists $(\pi, w) \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$ with $(\pi, w) \geq \left(\pi_{t-1}^{\max}, w_{t-1}\right)$.

Because $F(\pi; (A', L'))$ is strictly decreasing and continuous for each $(A', L') \in \mathcal{B}_t$, there exists some $\epsilon > 0$ such that for any $w' \in [w_{t-1} - \epsilon, w_{t-1}]$, there exists π' such that

[23] One possible solution to this indeterminacy is to consider some form of bargaining over distributions as an equilibrium selection mechanism. See, e.g., Cogliano et al. [8] and Yoshihara and Kaneko [45].

$(\pi', w') \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$. By assumption $w_{t-1} - p_{t-1}b \geqq 0$ is sufficiently small, and therefore we can consider $w' = 1 = p_{t-1}b$. By Lemma 3, there is a price vector $p' \in \triangle$ such that $p' = w'L^* [I - (1+\pi') A^*]^{-1} > \mathbf{0}$ and (A^*, L^*) is cost minimising at (p', w').

4. Noting that $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, let

$$\begin{aligned} x_t^\nu &= \frac{p'\omega_{t-1}^\nu}{p'\omega_{t-1}} A^{*-1}\omega_{t-1}, \; l_t^\nu = \frac{L^* A^{*-1}\omega_{t-1}}{N_t}, \; \delta_t^\nu = \mathbf{0}, \\ \omega_t^\nu &= \frac{p'x_t^\nu - w'L^*x_t^\nu + (w'-1)\, l_t^\nu}{(p' - L^*) A^{*-1}\omega_{t-1}} (I - bL^*) A^{*-1}\omega_{t-1}. \end{aligned}$$

Using the same argument as in step 2 of the proof of Theorem 3, it can then be proved that $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE with $p_t = p'$ and $w_t = w' = p_t b = 1$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

5. Let $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ be a CE such that $N_t > L^* x_t$. By Lemma 1(ii) and (D1c), $w_t = p_t b = 1$ follows from $N_t > L^* x_t$. By Theorem 2, $w_t \leqq w_{t-1}$ implies $\pi_t^{\max} > \pi_{t-1}^{\max}$. ■

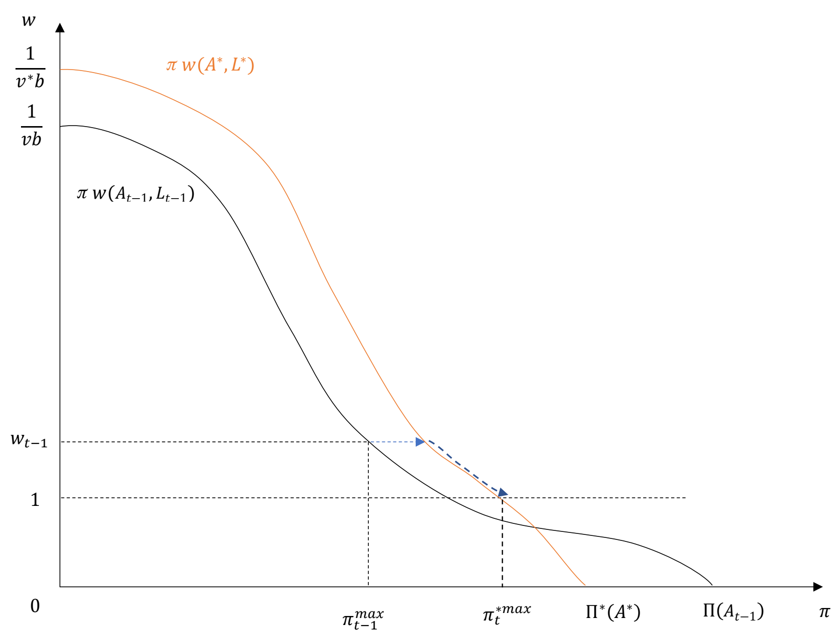

Figure 2

Figure 2: The effect of profitable, CU-LS innovations under the assumptions of Theorem 4. The new technique is adopted leading to an increase in the profit rate.

Theorem 4 shows that profitable, CU-LS innovations may indeed be adopted in equilibrium at t, provided w_{t-1} is sufficiently low. (This scenario is illustrated in Figure 2.) If this is not the case, however, new techniques may not be adopted.

To see this, suppose the economy is at a PCE with $N_{t-1} = L_{t-1}A_{t-1}^{-1}\omega_{t-2}$ at $t-1$ and let (A^*, L^*) be CU-LS, labour inelastic, and profitable. Then, $N_t > L^* A^{*-1}\omega_{t-1}$, as shown in step 2 of the proof of Theorem 4. Thus, if $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE with unemployment of labour, then it must be $w_t = p_t b = 1$ by Theorem 1(ii). Yet, while (A^*, L^*) yields higher profits than (A_{t-1}, L_{t-1}) in a neighbourhood of (p_{t-1}, w_{t-1}), it is not necessarily optimal at (p_t, w_t) if $w_t = 1$ is much lower than w_{t-1}. In this case, there may be a CE $\left((p_t, w_t); ((A_{t-1}, L_{t-1}); (x_t'^{\nu}; 1; \mathbf{0}); \omega_t'^{\nu})_{\nu \in \mathcal{N}_t}\right)$ with $A_{t-1}x_t' = \omega_{t-1}$ in which (A_{t-1}, L_{t-1}) is again optimal at (p_t, w_t). Figure 3 describes this situation.

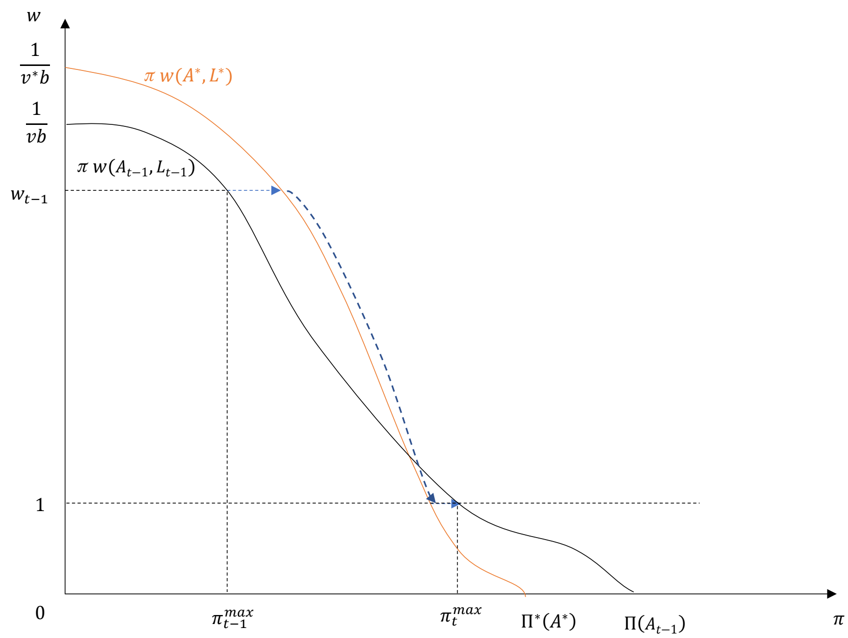

Figure 3

Figure 3: The effect of profitable, CU-LS innovations under the assumptions of Corollary 1 when $(\pi_t^{\max}, w_t = 1) \notin \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$. After the introduction of an innovation, equilibrium prices change and the old technique remains optimal.

The above argument can be summarised by the following corollary:

Corollary 1 *Suppose the economy is at a PCE with* $\pi_{t-1}^{\max} > 0$ *at* $t-1$. *Let* $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ *be profitable, CU-LS, and labour inelastic with* $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. *Then, there exists a CE* $\left((p_t, w_t), ((A^*, L^*); \xi_t^{\nu}; \omega_t^{\nu})_{\nu \in \mathcal{N}_t}\right)$ *with* $L^* x_t < N_t$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *if and only if* $(\pi_t^{\max}, w_t = 1) \in \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$.

5.3 Non-existence of equilibrium

Corollary 1 characterises the conditions under which what may be deemed a market failure occurs: if the condition in Corollary 1 is violated, there exists no equilibrium with positive profits in which a new technique is adopted *even if* it is profitable and increases labour productivity. Indeed, in this case localised innovations may cause an even deeper failure and disrupt the functioning of the economy in a more surprising and counterintuitive way: technical progress may cause the economy to reach *no* equilibrium at t.

Proposition 2 *There exist two economies,* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ *at* $t-1$ *and* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *at* t, *which together satisfy the following properties:*

(1) there exists a PCE $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu})_{\nu \in \mathcal{N}_{t-1}}\right)$ *for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ *such that* $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} > 1$;

(2) *there exists a profitable and CU-LS innovation* $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \setminus \mathcal{B}_{t-1}$ *with* $L^* A^{*-1}\omega_{t-1} < N_t$, *such that there exists no CE for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

Proof: Let $N_t \equiv 6$, $\omega_{t-1} \equiv \begin{pmatrix} 1 \\ 2.5 \end{pmatrix}$, $b \equiv \begin{pmatrix} 0.001 \\ 0.09 \end{pmatrix}$, $\mathcal{B}_{t-1} \equiv \{(A, L), (A^{**}, L^{**})\}$ and $\mathcal{B}_t \equiv \{(A, L), (A^*, L^*), (A^{**}, L^{**})\}$, where

$$\begin{aligned} (A, L) &\equiv \left(\begin{bmatrix} 0.085 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, (0.56, \ 0.4) \right), \\ (A^*, L^*) &\equiv \left(\begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, (0.4, \ 0.4) \right), \text{ and} \\ (A^{**}, L^{**}) &\equiv \left(\begin{bmatrix} 0.05 & 0.05 \\ 0.2 & 0.3 \end{bmatrix}, (0.4, \ 0.45) \right). \end{aligned}$$

All techniques in $\mathcal{B}_t$ are productive and indecomposable. Furthermore, $(A, L), (A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$ and $(A^{**}, L^{**}) \in \mathcal{B}_{t-1} \setminus \mathcal{B}_t(\omega_{t-1}, b)$ with $LA^{-1}\omega_{t-1} = N_t$, $L^* A^{*-1}\omega_{t-1} < N_t$, $L^{**}A^{**-1}\omega_{t-1} > N_t$, $L^* A^{*-1} \geq \mathbf{0}$, and $A^{**-1}\omega_{t-1} = (35, -15) \ngeq \mathbf{0}$. Finally, $A^* \geq A$ and $L^* \leq L$, with $A_1^* \geq A_1$ and $L_1^* < L_1$, $v^{**}b > vb > v^* b$, and $\Pi(A^{**}) > \Pi(A) > \Pi(A^*)$.[24]

In this economy, the WPFs at $t-1$ and t are depicted in Figure 4.[25]

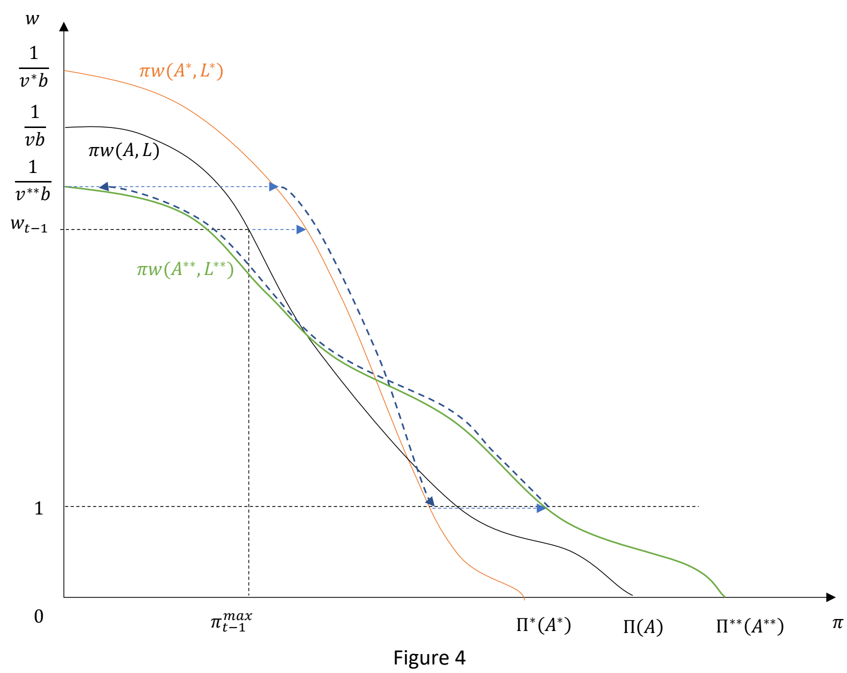

Figure 4: The WPFs corresponding to $\mathcal{B}_{t-1}$ and $\mathcal{B}_t$. Starting from a PCE at $t-1$ with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > 1$ no equilibrium can be reached at t.

Observe that $\pi w(A, L)$ is part of $\pi w(\mathcal{B}_{t-1})$ (around the combination of the highest wage rate and zero profit rate, as $\frac{1}{v^* b} > \frac{1}{vb} > \frac{1}{v^{**}b}$) but not of $\pi w(\mathcal{B}_t)$. Moreover, $\pi w(A^{**}, L^{**})$ is part of $\pi w(\mathcal{B}_t)$ at $w = 1$ and for wage rates sufficiently close to one.

[24] All of these claims can be directly verified. (See the Mathematica notebook in the online Addendum.)

[25] Figure 4 is not drawn to scale for visual clarity.

Part (1). Given ω_{t-1}, let $N_{t-1} = \frac{L\omega_{t-1}}{(1-Lb)}$ and $\omega_{t-2} = A\omega_{t-1} + AbN_{t-1} > \mathbf{0}$. Moreover, let $x_{t-1} = \omega_{t-1} + bN_{t-1} > \mathbf{0}$. Then, noting that the condition in Theorem 8 in the Appendix is satisfied, it is easy to check that a PCE exists for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ such that $(A_{t-1}, L_{t-1}) = (A, L)$ provided (p_{t-1}, w_{t-1}) is such that $w_{t-1} \in \left[\frac{1}{vb} - \varepsilon, \frac{1}{vb}\right)$ for a sufficiently small $\varepsilon > 0$ and $\left(\pi_{t-1}^{\max}, w_{t-1}\right) \in \pi w\left(\mathcal{B}_{t-1}\right) \cap \pi w\left(A, L\right)$.[26]

Part (2). Because $\pi w\left(A^*, L^*\right)$ is part of $\pi w\left(\mathcal{B}_t\right)$ for wage levels close to $\frac{1}{vb}$, (A^*, L^*) is profitable relative to $(A_{t-1}, L_{t-1}) = (A, L)$ by Lemma 4.[27] Suppose, ad absurdum, that a CE exists for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with prices (p_t, w_t). By Lemma 3, as $\pi w\left(A, L\right) \cap \pi w\left(\mathcal{B}_t\right) = \varnothing$, (A, L) will never be chosen.

Suppose $(A_t, L_t) = (A^*, L^*)$. By Theorem 1(ii), $L^* A^{*-1}\omega_{t-1} < N_t$ implies $w_t = 1$. However, by Lemma 3, (A^*, L^*) is not optimal at $(p_t, w_t = 1)$, yielding a contradiction.

Suppose $(A_t, L_t) = (A^{**}, L^{**})$. By Theorem 1(iii), $L^{**} A^{**-1}\omega_{t-1} > N_t$ implies $\pi_t^{*\max} = 0$. However, by Lemma 3, (A^{**}, L^{**}) is not optimal at (p_t, w_t) with $\pi_t^{\max} = 0$, as $\frac{1}{v^{**}b} < \frac{1}{vb} < \frac{1}{v^* b}$, yielding a contradiction.

Therefore, if a CE exists at t, then agents must activate a convex combination of (A^*, L^*) and (A^{**}, L^{**}), and be indifferent between the two techniques. Hence, in equilibrium $(\pi_t^{\max}, w_t)$ must be at the intersection of $\pi w\left(A^*, L^*\right)$ and $\pi w\left(A^{**}, L^{**}\right)$ on $\pi w\left(\mathcal{B}_t\right)$ in Figure 4, where $\pi_t^{\max} > 0$ and $w_t > 1$. By Theorem 1(i), if (p_t, w_t) with $(\pi_t^{\max}, w_t) > (0, 1)$ is part of a CE, then there must exist $x^*, x^{**} \in \mathbb{R}^2_+$ such that $A^* x^* + A^{**} x^{**} = \omega_{t-1}$, $L^* x^* + L^{**} x^{**} = N_t$, and $x^* + x^{**} \geqq N_t b > \mathbf{0}$.

We show that such x^*, x^{**} do not exist. First, as $L^* A^{*-1}\omega_{t-1} < N_t$, there is no $x^* \in \mathbb{R}^2_+$ such that $A^* x^* = \omega_{t-1}$ and $L^* x^* = N_t$. Then, by the Minkowski-Farkas Lemma (Gale, [15], p. 44, Theorem 2.6), there exists $(p'', w'') \in \mathbb{R}^{2+1}$ such that $p'' A^* + w'' L^* \geqq \mathbf{0}$ and $p''\omega_{t-1} + w'' N_t < 0$. Indeed, if $(p'', w'') \equiv ((0.5, 0.5), -0.3)$, then

$$p'' A^* + w'' L^* = (0.5, 0.5)\begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} - 0.3 \cdot (0.4, 0.4) = (0.03, 0.08) > \mathbf{0},$$

$$p''\omega_{t-1} + w'' N_t = (0.5, 0.5)\begin{pmatrix} 1 \\ 2.5 \end{pmatrix} - 0.3 \cdot 6 = 1.75 - 1.8 < 0.$$

Moreover, we have:

$$p'' A^{**} + w'' L^{**} = (0.5, 0.5)\begin{bmatrix} 0.05 & 0.05 \\ 0.2 & 0.3 \end{bmatrix} - 0.3 \cdot (0.4, 0.45) = (0.005, 0.04) > \mathbf{0}.$$

In other words, there exists $(p'', w'') \in \mathbb{R}^{2+1}$ such that $p'' A^* + w'' L^* \geqq \mathbf{0}$, $p'' A^{**} + w'' L^{**} \geqq \mathbf{0}$, and $p''\omega_{t-1} + w'' N_t < 0$. Then, by the Minkowski-Farkas Lemma (Gale, [15], p. 44, Theorem 2.6), there is no $x^*, x^{**} \in \mathbb{R}^2_+$ such that

$$\begin{bmatrix} A^* & A^{**} \\ L^* & L^{**} \end{bmatrix}\begin{pmatrix} x^* \\ x^{**} \end{pmatrix} = \begin{pmatrix} \omega_{t-1} \\ N_t \end{pmatrix}.$$

Hence, no convex combination of (A^*, L^*) and (A^{**}, L^{**}) can be activated at a CE in t.

In summary, there exists no CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. ■

[26] At $t-1$, agents can be assigned actions as in step 2 of the proof of Theorem 3.

[27] This claim can also be proved directly.

6 The falling profit rate

In the previous section, we have shown that – once the general equilibrium effects of technical progress are taken into account – the distributive effects of innovations are not obvious. Absent a significant shift in bargaining power towards labour, however, innovations – and especially labour saving ones – tend to increase profits. These results would seem to confirm the main intuition of OT and provide yet another obituary for Marx's theory of the falling profit rate. In this section, we show that, at a general level, this conclusion is unwarranted – or at least needs to be qualified – and some profitable innovations may indeed lead to a decrease in the equilibrium profit rate.

Our first result characterises the conditions under which profitable CS-LU innovations lead to a falling profit rate.

Theorem 5 *Suppose the economy is at a PCE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$ at $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ be profitable, CS-LU, and labour inelastic with $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. The following statements are equivalent:*

(1) there exists a CE $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$;

(2) $\pi_t^{\max} = 0$ at any CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ in which (A^*, L^*) is adopted;

(3) $(A^*, L^*) \in \arg\min_{(A',L')\in\mathcal{B}_t} L'(I - A')^{-1} b$.

Proof: ((1) ⇒ (2)) Suppose, ad absurdum, that $\pi_t^{\max} > 0$. By Lemma 1, $p_t A^* x_t^\nu = p_t \omega_{t-1}^\nu$, all $\nu \in \mathcal{N}_t$ and by Lemma 2, $p_t > \mathbf{0}$. Hence by (D1b), $A^* x_t = \omega_{t-1}$ and, noting that $L_{t-1}A_{t-1}^{-1}\omega_{t-1} = N_t$ by Proposition 1, it follows that $L_{t-1}A_{t-1}^{-1}A^* x_t = N_t$. By (D1c), and noting that $L^* \geq L_{t-1}$ and $x_t > \mathbf{0}$, it follows that $N_t \geqq L^* x_t > L_{t-1}x_t$. Therefore $L_{t-1}A_{t-1}^{-1}(A^* - A_{t-1}) x_t > 0$. Because the innovation is labour inelastic, $(L^* - L_{t-1}) x_t > L_{t-1}A_{t-1}^{-1}(A^* - A_{t-1}) x_t$, which implies $L^* x_t > N_t$, in contradiction with (D1c).

((3) ⇒ (1)) Let $(A^*, L^*) \in \arg\min_{(A',L')\in\mathcal{B}_t} L'(I - A')^{-1} b$. Then $\frac{1}{v'b} \leqq \frac{1}{v^*b}$ holds for all $(A', L') \in \mathcal{B}_t$ and by Lemma 3, (A^*, L^*) is optimal at $p_t = w_t v^* > \mathbf{0}$ and $w_t = \frac{1}{v^*b}$.

Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, let $x_t \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$. The argument used in the proof of ((1) ⇒ (2)) can be easily adapted to show that $N_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1} < L^* A^{*-1}\omega_{t-1}$. Then, let $k \equiv \frac{N_t}{L^* A^{*-1}\omega_{t-1}} < 1$. For all $\nu \in \mathcal{N}_t$, let

$$x_t^\nu = \frac{p_t \omega_{t-1}^\nu}{p_t \omega_{t-1}} k A^{*-1}\omega_{t-1} \geqq \mathbf{0}, \quad \delta_t^\nu = \frac{p_t \omega_{t-1}^\nu}{p_t \omega_{t-1}} (1-k)\,\omega_{t-1} \geqq \mathbf{0}, \quad l_t^\nu = 1,$$

$$\omega_t^\nu = \frac{p_t x_t^\nu - w_t L^* x_t^\nu + w_t - 1 + p_t \delta_t^\nu}{\left[k (p_t - L^*) A^{*-1}\omega_{t-1} + (1-k)\, p_t \omega_{t-1}\right]} \left[k (I - bL^*) A^{*-1}\omega_{t-1} + (1-k)\,\omega_{t-1}\right] \geq \mathbf{0}.$$

Using the same argument as in step 2 of the proof of Theorem 3, it can then be proved that $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

((2) ⇒ (3)) Let $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE with $\pi_t^{\max} = 0$. By Lemma 2, $p_t = p_t A^* + w_t L^*$ and therefore $p_t = w_t v^*$ and $w_t = \frac{1}{v^*b}$. Suppose, contrary to the statement, that for some $(A', L') \in \mathcal{B}_t$, $\frac{1}{v'b} > \frac{1}{v^*b}$. Then, $\left(\pi_t^{\max} = 0, w_t = \frac{1}{v^*b}\right) \notin \pi w(\mathcal{B}_t)$ and by Lemma 3, (A^*, L^*) is not optimal, a contradiction. ■

Theorem 5 shows the existence of localised innovations that are profitable from the viewpoint of an individual producer but which, if adopted universally, lead the equilibrium profit rate to fall. From a broad theoretical perspective, this result contradicts OT and may therefore be dubbed the *Anti-Okishio Theorem*. It identifies a scenario in which individually rational actions lead to collectively suboptimal outcomes, an intuition which is at the core of Marx's theory of technical change.[28]

How robust is the insight of Theorem 5? Does the equilibrium profit rate fall as a result of profitable, CS-LU innovations if condition (3) in Theorem 5 is not satisfied, or – more strongly – if an innovation is regressive? This is not obvious. It can be shown that if an innovation is CS-LU and regressive, then the new technique will not be adopted in equilibrium, even if it is profitable.[29] In this case, either an equilibrium emerges in which an old technique is adopted, or no equilibrium exists – as in the case discussed in section 5.3. Theorem 6 addresses the first scenario.

Theorem 6 *Suppose the economy is at a PCE with* $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} > p_{t-1}b$ *at* $t-1$. *Let* $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ *be profitable, CS-LU, and regressive. Let* $\{(A^{**}, L^{**})\} = \left[\arg\min_{(A',L')\in\mathcal{B}_{t-1}} L'(I-A')^{-1}b\right]$ *be such that* $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$. *Then, there exists a CE* $\left((p_t, w_t), ((A^{**}, L^{**}); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ *with* $\pi_t^{\max} < \pi_{t-1}^{\max}$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *whenever* $N_t \leqq L^{**}A^{**-1}\omega_{t-1}$. *Moreover, for any CE with* $\pi_t^{\max} < \pi_{t-1}^{\max}$ *in which* (A^{**}, L^{**}) *is adopted for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, $N_t = L^{**}A^{**-1}(\omega_{t-1} - \delta)$ *holds for some* $\delta \geqq \mathbf{0}$.

Proof: 1. Let $N_t \leqq L^{**}A^{**-1}\omega_{t-1}$. As (A^*, L^*) is regressive, it follows that $vb < v^*b$. Thus, by the definition of (A^{**}, L^{**}), $v^{**}b < v^*b$ and $\left(0, \frac{1}{v^{**}b}\right) \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$. Because $F(\pi; (A^{**}, L^{**}))$ is strictly decreasing and continuous, there exists $\varepsilon > 0$ such that $\frac{1}{v^{**}b} - \varepsilon > w_{t-1}$ and for all $w' \in \left[\frac{1}{v^{**}b} - \varepsilon, \frac{1}{v^{**}b}\right]$ there exists $\pi' \geqq 0$ such that $(\pi', w') \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$. By Lemma 3, for all such (π', w'), there exists $p' \in \Delta$ with $p' = w'L^{**}\left[I - (1+\pi')A^{**}\right]^{-1} > \mathbf{0}$, such that (A^{**}, L^{**}) is optimal at (p', w').

Suppose $N_t = L^{**}A^{**-1}\omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then the same argument as in step 2 of the proof of Theorem 3 shows that there is a CE $\left((p_t, w_t), ((A^{**}, L^{**}); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ with $(p_t, w_t) = (p', w')$ and $\pi_t^{\max} = \pi'$ for all of the above mentioned $(\pi', w') \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$. By construction, $w' > w_{t-1}$ implies $\pi' < \pi_{t-1}^{max}$.

Suppose $N_t < L^{**}A^{**-1}\omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then the same argument as in the proof of Theorem 5 ((3) $\Rightarrow$ (1)) shows that there is a CE $\left((p_t, w_t), ((A^{**}, L^{**}); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ with $(p_t, w_t) = \left(\frac{1}{v^{**}b}v^{**}, \frac{1}{v^{**}b}\right)$ and $\pi_t^{\max} = 0 < \pi_{t-1}^{\max}$.

2. Let $\left((p_t, w_t), ((A^{**}, L^{**}); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $\pi_t^{\max} < \pi_{t-1}^{\max}$. By Lemma 2 and (D1b), it follows that $A^{**}x_t + \delta_t = \omega_{t-1}$ (or equivalently $x_t = A^{**-1}(\omega_{t-1} - \delta_t)$). Suppose, contrary to the statement, $N_t \neq L^{**}A^{**-1}(\omega_{t-1} - \delta)$ for all $\delta \geqq \mathbf{0}$. Then, $N_t \neq L^{**}x_t$ as $x_t = A^{**-1}(\omega_{t-1} - \delta_t)$, which implies $L^{**}x_t < N_t$ by (D1c). Thus, at the CE it would be $w_t = 1 < w_{t-1}$, which contradicts the assumption that $\pi_t^{\max} < \pi_{t-1}^{\max}$, by Lemma 3 and the properties of the WPF. ■

Theorem 6 suggests that the insight of Theorem 5 is indeed robust: there exist a range of scenarios in which the emergence of individually profitable innovations leads to a decline in the equilibrium profit rate. The mechanism highlighted in Theorem 6, however,

[28] Setting aside the empirically less relevant case of innovations that shift the *whole* WPF out, it can be shown that the conditions in Theorem 5 describe a scenario characterised by so-called *re-switching* and *reverse capital deepening*; see Kurz and Salvadori [24]. This suggests that there may be some interesting and perhaps surprising connections between the theory of the falling profit rate and some central insights of classical capital theory. (For a discussion, see the Addendum.)

[29] For a formal proof, see Theorem A1(ii) in the Addendum.

is rather different and the result provides an original perspective on the debates on the falling rate of profit. For it shows that technical progress may indeed lead to a decline in profitability because of the general equilibrium effects of localised innovations even though, unlike in Theorem 5, *the new technique is not adopted in equilibrium.*

The main effect of localised innovations, in Theorem 6, is to disrupt the status quo. The appearance of the new, profitable technique (A^*, L^*) leads agents to abandon old production methods, moving the economy away from equilibrium. The innovation, however, is regressive and is not optimal at any CE. One may imagine a process of trial and error in which the economy deviates from the status quo prices (p_{t-1}, w_{t-1}) and eventually settles on another equilibrium in which a previously suboptimal technique, (A^{**}, L^{**}), is adopted.[30] If capital becomes relatively abundant $(N_t < L^{**}A^{**-1}\omega_{t-1})$, then the profit rate falls to zero. However, perhaps more surprisingly, Theorem 6 proves that there is a decrease in the equilibrium profit rate even if the economy moves to an equilibrium with full employment of labour and capital $(N_t = L^{**}A^{**-1}\omega_{t-1})$ and strictly positive profits.

Two additional comments are in order. First, if the condition, $N_t = L^{**}A^{**-1}(\omega_{t-1} - \delta)$ for some $\delta \geqq \mathbf{0}$, in Theorem 6 is violated, and so there is an excess supply of labour with (A^{**}, L^{**}), then using a similar argument as in section 5.3 it can be shown that there may be no CE in the economy. When technical progress is localised, the non-existence of equilibrium may be a pervasive problem.

Second, Theorems 5 and 6 hold under the assumption of full employment of labour at the PCE in $t-1$. What if, instead, there is a sufficiently big industrial reserve army of the unemployed? It can be shown that if $N_{t-1} > L_{t-1}x_{t-1}$ and $A_{t-1}x_{t-1} = \omega_{t-2}$ in $t-1$, then a profitable CS-LU innovation will be adopted in equilibrium with unemployment of labour, and lead to an increase in the profit rate, provided it is gradual.[31] This scenario could obtain, for example, in a developing economy in which aggregate capital is still low relative to the labour force.

Theorems 5 and 6 prove that CS-LU innovations may cause the profit rate to fall. Is this the *only* scenario that may lead to a decrease in the profit rate? Not really. Theorem 7 proves that if general equilibrium effects are considered, then the profit rate may fall even in the standard case of CU-LS innovation.[32]

Theorem 7 *Suppose the economy is at a PCE with $\pi_{t-1}^{\max} > 0$ at $t-1$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ be profitable and CU-LS with $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$. The following statements are equivalent:*

(1) there exists a CE $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$;

(2) $\pi_t^{\max} = 0$ for any CE in which (A^*, L^*) is adopted for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$;

(3) there exists $x > \mathbf{0}$ such that $(I - bL^*)x \geqq A^*x - \omega_{t-1}$ with $A^*x \leq \omega_{t-1}$ and $L^*x = N_t$, and moreover, $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L'(I - A')^{-1} b$.

[30] Interestingly, although (A^*, L^*) is CS-LU, the production technique that is actually adopted in equilibrium is *more* capital intensive than the original technique (A, L), where the value of capital is evaluated using the price vector corresponding to the switching point of these techniques on $\pi w(\mathcal{B}_{t-1})$.

[31] For a formal statement, see the Addendum.

[32] The proof of Theorem 7 is similar to that of Theorem 5 – noting that $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$ leads to $\pi_t^{*\max} = 0$ – and is therefore omitted. (See the Addendum.)

Indeed, any profitable CU-LS innovation will *always* lead the profit rate to fall to zero in equilibrium, under condition (3) of Theorem 7 with $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$. It is not difficult to find an economy in which this condition is non-vacuous.

7 Conclusions

Our results paint a much more complex picture of the effects of innovations than in the standard approach. There is no obvious relation between ex-ante profitable innovations and the (functional) distribution of income that emerges in equilibrium after localised technical change is implemented. If technical change leads to an equilibrium with full employment of productive factors, the distribution of income is undetermined, and it is possible for either the profit rate or the wage rate to decrease. But with localised innovations there is no guarantee that the equilibrium will be characterised by full employment. Furthermore, a localised innovation that is individually profitable at status quo prices does not necessarily yield an increase in profitability: after it is universally adopted, a change in equilibrium prices may occur eventually leading the profit rate to decrease.

Methodologically, our analysis suggests that the distributive effects of technical progress cannot be fully understood in models that do not capture the dialectic between individual choices and aggregate outcomes, and the complex network of effects induced by localised technical change. A general equilibrium approach to technical change allows us to model some aspects of the Schumpeterian process of creative destruction. Even though they affect *only* the production techniques currently in use, – unlike in the standard analysis of technical progress, – localised innovations disrupt the status quo and move an economy away from its original equilibrium. Indeed, they may even cause the disappearance of *all* equilibria and lead the economy to a persistent disequilibrium dynamics. This methodological insight is, we believe, robust and our theoretical approach provides a rich framework for the analysis of innovations.

In closing, we briefly mention three possible extensions of our analysis. First, we have focused only on *process* innovations – new ways of combining inputs in the production of a given set of goods. It would be interesting to investigate the distributive effects of *product* innovations – the invention of new goods. Second, given our focus on the effect of the appearance of innovations on wages and profits, we have not explicitly modelled the process of discovering new techniques. Yet, from the general equilibrium perspective adopted in our paper, it would be interesting to endogenise R&D activities and investment (for a preliminary analysis in an one-good model, see Cogliano et al. [8]). Finally, we have followed the classical literature on localised innovations by focusing on economies with homogeneous labour. It would be worth extending our analysis to more complex models with heterogeneous labour inputs (for a discussion of classical models with multiple non-reproducible factors, see, e.g., Kurz and Salvadori [24] and Ekeland and Guesnerie [10]): in addition to allowing for a richer picture of production processes, and of innovations, this would also provide a more nuanced analysis of the distributive effects of innovations (by distinguishing, for example, between high-skilled and low-skilled workers).

References

[1] Acemoglu, D., 2009. *Introduction to Modern Economic Growth.* Princeton University Press, Princeton.

[2] Acemoglu, D., 2015. Localised and Biased Technologies. *The Economic Journal* 125, 443-463.

[3] Atkinson, A. B. and Stiglitz, J. E., 1969. A New View of Technological Change. *The Economic Journal* 79, 573-578.

[4] Bidard, C., 1999. Fixed capital and internal rate of return. *Journal of Mathematical Economics* 31, 523-544.

[5] Blanchard, O., 2022. *Fiscal Policy under Low Interest Rates.* MIT Press, Cambridge MA.

[6] Boppart, T., 2014. Structural Change and the Kaldor Facts in a Growth Model with Relative Price Effects and Non-Gorman Preferences, *Econometrica* 82, 2167-2196.

[7] Caselli, F., 2005. Accounting for Cross-Country Income Difference. in: Aghion, P., Durlauf, S. (eds.), *Handbook of Economic Growth, Vol. 1A*. North Holland, Amsterdam, pp. 679-741.

[8] Cogliano, J., Veneziani, R., Yoshihara, N., 2016. The Dynamic of Exploitation and Class in Accumulation Economies. *Metroeconomica* 67, 242-290.

[9] Dana, R.-A., Florenzano, M., Le Van, C., Levy, D., 1989. Production prices and general equilibrium prices. *Journal of Mathematical Economics* 18, 263-280.

[10] Ekeland, I., Guesnerie, R., 2010. The geometry of global production and factor price equalisation. *Journal of Mathematical Economics* 46, 666-690.

[11] Flaschel, P., Franke, R., Veneziani, R., 2013. Labor Productivity and the Law of Decreasing Labor Content. *Cambridge Journal of Economics* 37, 379-402.

[12] Franke, R., 1999. Technical change and a falling wage share if profits are maintained. *Metroeconomica* 50, 35-53.

[13] Freni, G., Gozzi, F., Salvadori, N., 2006. Existence of optimal strategies in linear multisector models. *Economic Theory* 29, 25-48.

[14] Galanis, G., Veneziani, R., Yoshihara, N., 2019. The dynamics of inequalities and unequal exchange of labor in intertemporal linear economies. *Journal of Economic Dynamics and Control* 100, 29-46.

[15] Gale, D., 1960. *The Theory of Linear Economic Models.* Chicago University Press, Chicago.

[16] Grandmont, J. M., 1977. Temporary General Equilibrium Theory. *Econometrica* 45, 535-572.

[17] Hahn, F. H., 1971. Equilibrium with Transaction Costs. *Econometrica* 39, 417-439.

[18] Hicks, J., 1946. *Value and Capital*, 2nd ed. Oxford: Clarendon Press.

[19] Jones, C. I., 2005. The Shape of Production Functions and the Direction of Technical Change. *Quarterly Journal of Economics* 120, 517-549.

[20] Jordà, O., Knoll, K., Kuvshinov, D., Schularick, M., Taylor, A. M., 2019. The rate of return on everything, 1870-2015. *Quarterly Journal of Economics* 134, 1225-1298.

[21] Kiedrowski, R., 2018. Profit rates equalization and balanced growth in a multi-sector model of classical competition. *Journal of Mathematical Economics* 77, 39-53.

[22] Kongsamut, P., Rebelo, S., Xie, D., 2001. Beyond Balanced Growth. *Review of Economic Studies* 68, 869-882.

[23] Kurose, K., 2021. Models of Structural Change and Kaldor's Facts. *Structural Change and Economic Dynamics* 58, 267-277.

[24] Kurz, D.K., Salvadori, N., 1995. *Theory of Production: A Long-Period Analysis*. Cambridge University Press, Cambridge.

[25] Mandler, M., 1999. Sraffian Indeterminacy in General Equilibrium. *Review of Economic Studies*, 66, 693-711.

[26] Marx, K., 1959 [1894]. *Capital*, Vol.III. Lawrence & Wishart, London.

[27] Morishima, M., 1964. *Equilibrium, Stability and Growth*. Oxford: Clarendon Press.

[28] Neumann, von J., 1945. A Model of General Economic Equilibrium, *Review of Economic Studies* 13, 1-9.

[29] Ngai, L., Pissarides, C., 2007. Structural Change in a Multisector Model of Growth. *American Economic Review* 97, 429-443.

[30] Okishio, N., 1961. Technical Change and the Rate of Profit. *Kobe University Economic Review* 7, 86-99.

[31] Piketty, T., 2014. *Capital in the Twenty-First Century*, Harvard University Press, Cambridge, MA.

[32] Radner, R., 1972. Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets. *Econometrica* 40, 289-303.

[33] Roemer, J.E., 1977. Technical change and the 'tendency of the rate of profit to fall'. *Journal of Economic Theory* 16, 403-424.

[34] Roemer, J.E., 1979. Continuing controversy on the falling rate of profit: fixed capital and other issues. *Cambridge Journal of Economics* 3, 379-398.

[35] Roemer, J.E., 1980. A General Equilibrium Approach to Marxian Economics. *Econometrica* 48, 505-535.

[36] Roemer, J.E., 1980. Innovation, rates of profit, and uniqueness of von Neumann prices. *Journal of Economic Theory* 22, 451-464.

[37] Roemer, J.E., 1982. *A General Theory of Exploitation and Class.* Harvard University Press, Cambridge, MA.

[38] Schumpeter, J.A., 1934. *The Theory of Economic Development.* Harvard University Press, Cambridge, MA.

[39] Sraffa, P., 1960. *Production of Commodities by Means of Commodities*: *Prelude to a Critique of Economic Theory*, Cambridge University Press, Cambridge.

[40] Summers, L.H., 2014. US economic prospects: secular stagnation, hysteresis, and the zero lower bound. *Business economics* 49, 65-73.

[41] Summers, L.H., 2016. The Age of Secular Stagnation: What It Is and What to Do About It. *Foreign Affairs* 95, 2-9.

[42] Takahashi, H., 2008. Optimal balanced growth in a general multi-sector endogenous growth model with constant returns. *Economic Theory* 37, 31-49.

[43] Veneziani, R., 2007. Exploitation and time. *Journal of Economic Theory* 132, 189-207.

[44] Veneziani, R., Yoshihara, N., 2017. Globalisation and inequality in a dynamic economy: an axiomatic analysis of unequal exchange. *Social Choice and Welfare* 49, 445-468.

[45] Yoshihara, N., Kaneko, S., 2016. On the Existence and Characterization of Unequal Exchange in the Free Trade Equilibrium. *Metroeconomica* 67, 210-241.

[46] Yoshihara, N., Kwak, S-H., 2023. Sraffian indeterminacy of steady-state equilibria in the Walrasian general equilibrium framework, working paper, University of Massachusetts, Amherst.

A The existence of a PCE at $t-1$

In this Appendix, we provide a complete characterisation of the conditions for the existence of a PCE for an economy $E(\mathcal{N}_{t-1};\mathcal{B}_{t-1};b;\Omega_{t-2})$ in period $t-1$. Let

$$\overline{\mathcal{B}}_{t-1} \equiv \left\{(A,L)\in\mathcal{B}_{t-1} \mid \exists\,(\pi,w)\in\pi w\,(A,L)\cap\pi w\,(\mathcal{B}_{t-1}) \text{ s.t. } (\pi,w)>(0,1)\right\}.$$

A.1 Full employment

Given an economy $E(\mathcal{N}_{t-1};\mathcal{B}_{t-1};b;\Omega_{t-2})$ at period $t-1$ and given $\mathcal{N}_t$, let

$$C_{t-1}\equiv \left\{\omega\in\mathbb{R}^n_+ \mid \exists x>\mathbf{0} \;\&\; (A,L)\in\overline{\mathcal{B}}_{t-1}\text{:} \begin{array}{c} Ax=\omega,\ (I-bL)\,x\geq\mathbf{0},\ Lx=N_{t-1}, \\ A^{-1}\,(x-N_{t-1}b)>\mathbf{0},\ LA^{-1}\,(x-N_{t-1}b)=N_t \end{array}\right\}.$$

Theorem 8 proves that $\omega_{t-2}\in C_{t-1}$ is the necessary and sufficient condition for the existence of a PCE with full employment of all productive factors.

Theorem 8 *Consider an economy $E(\mathcal{N}_{t-1};\mathcal{B}_{t-1};b;\Omega_{t-2})$ at $t-1$ and a set $\mathcal{N}_t$. There exists a PCE $\left((p_{t-1},w_{t-1}),\left((A_{t-1},L_{t-1});\xi^\nu_{t-1};\omega^\nu_{t-1}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ such that $\pi^{max}_{t-1}>0$ and $w_{t-1}>1$ if and only if $\omega_{t-2}\in C_{t-1}$.*

Proof. ($\Rightarrow$) Let $\left((p_{t-1},w_{t-1}),\left((A_{t-1},L_{t-1});\xi^\nu_{t-1};\omega^\nu_{t-1}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ be a PCE such that $\pi^{max}_{t-1}>0$ and $w_{t-1}>1$. By Lemma 3, $(A_{t-1},L_{t-1})\in\overline{\mathcal{B}}_{t-1}$. By Lemmas 1-2, $L_{t-1}x_{t-1}=N_{t-1}$ and $A_{t-1}x_{t-1}=\omega_{t-2}$. By (D1d), $\omega_{t-1}=x_{t-1}-N_{t-1}b$. As the CE is persistent, $\omega_{t-1}\geq\mathbf{0}$ holds, which implies $x_{t-1}-N_{t-1}b=(I-bL)\,x_{t-1}\geq\mathbf{0}$. Moreover, by Proposition 1(i), $A^{-1}_{t-1}\omega_{t-1}>\mathbf{0}$ and $L_{t-1}A^{-1}_{t-1}\omega_{t-1}=N_t$, which imply $A^{-1}_{t-1}\,(x_{t-1}-N_{t-1}b)>\mathbf{0}$ and, $L_{t-1}A^{-1}_{t-1}\,(x_{t-1}-N_{t-1}b)=N_t$, respectively. In conclusion, $\omega_{t-2}\in C_{t-1}$.

($\Leftarrow$) Suppose $\omega_{t-2}\in C_{t-1}$. Then, there exist $x>\mathbf{0}$ and $(A,L)\in\overline{\mathcal{B}}_{t-1}$ such that $Ax=\omega_{t-2}$, $(I-bL)\,x\geq\mathbf{0}$, $Lx=N_{t-1}$, $A^{-1}\,(x-N_{t-1}b)>\mathbf{0}$, and $LA^{-1}\,(x-N_{t-1}b)=N_t$. As $(A,L)\in\overline{\mathcal{B}}_{t-1}$, then by Lemma 3, there exists $\left(\pi^{\max}_{t-1},w_{t-1}\right)>(0,1)$ such that for $p_{t-1}\equiv w_{t-1}L\left[I-\left(1+\pi^{\max}_{t-1}\right)A\right]^{-1}>\mathbf{0}$,

$$p_{t-1}=\left(1+\pi^{\max}_{t-1}\right)p_{t-1}A+w_{t-1}L\leqq\left(1+\pi^{\max}_{t-1}\right)p_{t-1}A'+w_{t-1}L' \text{ for any } (A',L')\in\mathcal{B}_{t-1}.$$

Then, an action profile $\left(\xi^\nu_{t-1};\omega^\nu_{t-1}\right)$ can be assigned to each $\nu\in\mathcal{N}_{t-1}$ as in step 2 of Theorem 3 such that $\left((p_{t-1},w_{t-1}),\left((A,L);\xi^\nu_{t-1};\omega^\nu_{t-1}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ is a CE with $\left(\pi^{\max}_{t-1},w_{t-1}\right)>(0,1)$. At this CE, $\sum_{\nu\in\mathcal{N}_{t-1}}x^\nu_{t-1}=x$ and $\omega_{t-1}=x-N_{t-1}b$. Therefore $A^{-1}\,(x-N_{t-1}b)>\mathbf{0}$ and $LA^{-1}\,(x-N_{t-1}b)=N_t$, imply, respectively, $A^{-1}\omega_{t-1}>\mathbf{0}$ and $LA^{-1}\omega_{t-1}=N_t$ and Proposition 1 implies that this CE is persistent. ■

Next, we prove that if a mild condition on population growth is satisfied, then the set C_{t-1} is well-defined. Let

$$\overline{\mathcal{B}}_{t-1}\,(N_{t-1},N_t)\equiv\left\{(A,L)\in\overline{\mathcal{B}}_{t-1}\mid\exists g_{(A,L)}>0: L\left[I-\left(1+g_{(A,L)}\right)A\right]^{-1}b=1 \;\&\; \frac{N_t}{N_{t-1}}=1+g_{(A,L)}\right\}.$$

Theorem 9 *Consider an economy $E(\mathcal{N}_{t-1};\mathcal{B}_{t-1};b;\Omega_{t-2})$ at $t-1$ and a set $\mathcal{N}_t$. If $\overline{\mathcal{B}}_{t-1}\,(N_{t-1},N_t)\neq\varnothing$, then $C_{t-1}\neq\varnothing$.*

Proof. As $\overline{\mathcal{B}}_{t-1}(N_{t-1}, N_t) \neq \varnothing$, let $(A, L) \in \overline{\mathcal{B}}_{t-1}(N_{t-1}, N_t)$. Let $(1+g) \equiv \frac{N_t}{N_{t-1}}$. Then, $L[I-(1+g)A]^{-1}b = 1$ holds. The last equation implies that there exists $\overline{p} \in \triangle$ such that $\overline{p} \equiv L[I-(1+g)A]^{-1} > \mathbf{0}$. Therefore, $\overline{p} = \overline{p}[(1+g)A + bL]$ holds, which implies that the Frobenius eigenvalue of the matrix $[(1+g)A + bL]$ is equal to 1 and is associated with the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$. (See Kurz and Salvadori [24]; Theorem A.3.5.) Then, there exists a Frobenius eigenvector $\overline{x} > \mathbf{0}$ such that $\overline{x} = [(1+g)A + bL]\overline{x}$ unique up to $L\overline{x} = N_{t-1}$. Then, $(1+g)A\overline{x} = \overline{x} - N_{t-1}b$ holds. As $(1+g)A\overline{x} > \mathbf{0}$ by the indecomposability of A and $\overline{x} > \mathbf{0}$, $\overline{x} - N_{t-1}b = (I - bL)\overline{x} > \mathbf{0}$. Moreover, $A^{-1}(\overline{x} - N_{t-1}b) = A^{-1}(1+g)A\overline{x} = (1+g)\overline{x} > \mathbf{0}$. Finally, $LA^{-1}(x - N_{t-1}b) = LA^{-1}(1+g)A\overline{x} = \frac{N_t}{N_{t-1}}N_{t-1}$. Thus, setting $\omega \equiv A\overline{x}$, we conclude that $\omega \in C_{t-1}$. ■

Remark 1: Remember that, for any $(A, L) \in \overline{\mathcal{B}}_{t-1}$, $1 > vb = L[I-A]^{-1}b$ holds by assumption. Moreover, $L[I-(1+\pi)A]^{-1}b = L\left[\sum_{k=0}^{\infty}((1+\pi)A)^k\right]b$ is strictly increasing at every $\pi \in [0, \Pi(A))$ and $\lim_{\pi \to \Pi(A)} L[I-(1+\pi)A]^{-1}b = \infty$. Thus, by the intermediate value theorem, there exists $g_{(A,L)} > 0$ such that $L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1}b = 1$.

A.2 Labour unemployment

Given an economy $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ in period $t-1$ and a set $\mathcal{N}_t$, let

$$\overline{\mathcal{B}}'_{t-1} \equiv \{(A, L) \in \mathcal{B}_{t-1} \mid \exists (\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_{t-1}) \text{ s.t. } \pi > 0 \ \& \ w = 1\},$$

$$C'_{t-1} \equiv \left\{\omega \in \mathbb{R}^n_+ \mid \exists x > \mathbf{0} \ \& \ (A, L) \in \overline{\mathcal{B}}^*_{t-1} \text{: } \begin{array}{c} Ax = \omega, \ (I - bL)x \geq \mathbf{0}, \ Lx < N_{t-1}, \\ A^{-1}(I - bL)x > \mathbf{0}, \ LA^{-1}(I - bL)x \leqq N_t \end{array}\right\}.$$

Theorem 10 shows that $\omega_{t-2} \in C'_{t-1}$ is necessary and sufficient for the existence of a PCE at $t-1$ with unemployment of labour.

Theorem 10 *Consider an economy $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ at $t-1$ and a set $\mathcal{N}_t$. There exists a PCE $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ such that $\pi^{max}_{t-1} > 0$ and $w_{t-1} = p_{t-1}b$ if and only if $\omega_{t-2} \in C'_{t-1}$.*

Proof. $(\Rightarrow)$ Let $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE such that $\pi^{max}_{t-1} > 0$ and $w_{t-1} = p_{t-1}b = 1$. By Lemma 3, $(A_{t-1}, L_{t-1}) \in \overline{\mathcal{B}}'_{t-1}$. By Lemmas 1-2, $A_{t-1}x_{t-1} = \omega_{t-2}$. By (D1d), $\omega_{t-1} = x_{t-1} - bL_{t-1}x_{t-1}$. As the CE is persistent, $\omega_{t-1} \geq \mathbf{0}$ holds, which implies $(I - bL_{t-1})x_{t-1} \geq \mathbf{0}$. Moreover, by Proposition 1(ii), $A^{-1}_{t-1}\omega_{t-1} > \mathbf{0}$ and $L_{t-1}A^{-1}_{t-1}\omega_{t-1} \leqq N_t$ imply $A^{-1}_{t-1}(x_{t-1} - bL_{t-1}x_{t-1}) > \mathbf{0}$ and $L_{t-1}A^{-1}_{t-1}(x_{t-1} - bL_{t-1}x_{t-1}) \leqq N_t$, respectively. In conclusion, $\omega_{t-2} \in C'_{t-1}$.

$(\Leftarrow)$ Let $\omega_{t-2} \in C'_{t-1}$ hold. Then, there exist $x > \mathbf{0}$ and $(A, L) \in \overline{\mathcal{B}}'_{t-1}$ such that $Ax = \omega_{t-2}$, $(I - bL)x \geq \mathbf{0}$, $Lx < N_{t-1}$, $A^{-1}(I - bL)x > \mathbf{0}$, and $LA^{-1}(I - bL)x \leqq N_t$. As $(A, L) \in \overline{\mathcal{B}}'_{t-1}$, then by Lemma 3, there exist $\pi^{\max}_{t-1} > 0$ and $w_{t-1} = 1$ such that for $p_{t-1} \equiv w_{t-1}L\left[I - \left(1 + \pi^{\max}_{t-1}\right)A\right]^{-1} > \mathbf{0}$,

$$p_{t-1} = \left(1 + \pi^{\max}_{t-1}\right)p_{t-1}A + w_{t-1}L \leqq \left(1 + \pi^{\max}_{t-1}\right)p_{t-1}A' + w_{t-1}L' \text{ for any } (A', L') \in \mathcal{B}_{t-1}.$$

Then, an action profile $\left(\xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)$ can be assigned to each $\nu \in \mathcal{N}_{t-1}$ as in step 4 of Theorem 4 such that $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ is a CE with $\pi^{\max}_{t-1} > 0$ and

$w_{t-1} = 1$. At this CE, $\sum_{\nu \in \mathcal{N}_{t-1}} x^{\nu}_{t-1} = x$ and $\omega_{t-1} = x - bLx$. Therefore, $A^{-1}(I - bL)x > \mathbf{0}$ and $LA^{-1}(x - bLx) \leqq N_t$ imply, respectively, $A^{-1}\omega_{t-1} > \mathbf{0}$ and $LA^{-1}\omega_{t-1} \leqq N_t$ and Proposition 1(ii) implies that this CE is persistent. ■

Next, we show that if a mild condition on population growth is satisfied, then C'_{t-1} is well-defined. Let

$$\overline{\mathcal{B}}'_{t-1}(N_{t-1}, N_t) \equiv \left\{ (A, L) \in \overline{\mathcal{B}}'_{t-1} \mid \exists g_{(A,L)} > 0 : L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} b = 1 \ \& \ \frac{N_t}{N_{t-1}} \geqq 1 + g_{(A,L)} \right\}.$$

Theorem 11 *Consider an economy $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ at $t-1$ and a set $\mathcal{N}_t$. If $\overline{\mathcal{B}}'_{t-1}(N_{t-1}, N_t) \neq \varnothing$, then $C'_{t-1} \neq \varnothing$.*

Proof. Let $(A, L) \in \overline{\mathcal{B}}'_{t-1}(N_{t-1}, N_t)$. Then, for some $g_{(A,L)} > 0$, $L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} b = 1$ holds, which implies that there exists $\overline{p} \in \triangle$ such that $\overline{p} \equiv L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} > \mathbf{0}$. Therefore, $\overline{p} = \overline{p}\left[\left(1 + g_{(A,L)}\right)A + bL\right]$ holds, and the Frobenius eigenvalue of the matrix $\left[\left(1 + g_{(A,L)}\right)A + bL\right]$ associated with the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$ is equal to one. Then, there exists the Frobenius eigenvector $\overline{x} > \mathbf{0}$ such that $\overline{x} = \left[\left(1 + g_{(A,L)}\right)A + bL\right]\overline{x}$ with $L\overline{x} < N_{t-1}$. Then, $\left(1 + g_{(A,L)}\right)A\overline{x} = \overline{x} - bL\overline{x} > \mathbf{0}$ holds by $\overline{x} > \mathbf{0}$ and the indecomposability of A. Moreover, $A^{-1}(I - bL)\overline{x} = A^{-1}\left(1 + g_{(A,L)}\right)A\overline{x} = \left(1 + g_{(A,L)}\right)\overline{x} > \mathbf{0}$ holds. Finally, we have $LA^{-1}(\overline{x} - bL\overline{x}) = LA^{-1}\left(1 + g_{(A,L)}\right)A\overline{x} = \left(1 + g_{(A,L)}\right)L\overline{x} < \left(1 + g_{(A,L)}\right)N_{t-1} \leqq N_t$ as $\frac{N_t}{N_{t-1}} \geqq 1 + g_{(A,L)}$ holds by $(A, L) \in \overline{\mathcal{B}}'_{t-1}(N_{t-1}, N_t)$. Thus, $LA^{-1}(\overline{x} - bL\overline{x}) < N_t$. Thus, setting $\omega \equiv A\overline{x}$, we conclude that $\omega \in C'_{t-1}$. ■

Remark 2: Using the same argument as in Remark 1, for any $(A, L) \in \overline{\mathcal{B}}'_{t-1}$, there exists $g_{(A,L)} > 0$ such that $L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} b = 1$. Therefore, if population growth, $\frac{N_t}{N_{t-1}}$, is sufficiently high, then the set $\overline{\mathcal{B}}'_{t-1}(N_{t-1}, N_t)$ is non-empty.

Addendum for the paper: "*The general equilibrium effects of localised technological progress: A Classical approach*"

Naoki Yoshihara* Roberto Veneziani†

May 1, 2023

Abstract

Section 1 contains the proof of Theorem 7 which is omitted in the paper. Section 2 analyse the effects of innovations on income distribution, under the assumption that the economy moves to a new equilibrium in which the new technique is actually adopted. Section 3 provides a discussion of the connections between the falling rate of profit and capital theory mentioned in section 6 of the paper. Section 4 provides a formal statement of the distributive implications of technical progress in developing economies.

*Department of Economics, University of Massachusetts Amherst, Crotty Hall, Amherst, MA, 01002, USA; The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-0004, Japan; and School of Management, Kochi University of Technology, Tosayamada, Kami-city, Kochi 782-8502, Japan. (nyoshihara@econs.umass.edu)

†School of Economics and Finance, Queen Mary University of London, Mile End Road, London E1 4NS, UK. (r.veneziani@qmul.ac.uk)

1 Proof of Theorem 7

Proof: ((2) $\Rightarrow$ (3)) Let $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ be a CE with $\pi_t^{\max} = 0$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. As $\pi_t^{\max} = 0$, it follows that $(p_t, w_t) = \left(\frac{1}{v^* b} v^*, \frac{1}{v^* b}\right)$ and $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$. Moreover, as $w_t > p_t b$, (D1c)-(D1d) together with Lemma 1 imply $x_t \equiv \sum_{\nu \in \mathcal{N}_t} x_t^\nu > \mathbf{0}$ and $L^* x_t = N_t$. Suppose, by way of contradiction, that $A^* x_t = \omega_{t-1}$. Because $(A^*, L^*) \notin \mathcal{B}_t (\omega_{t-1}, b)$, then $(I - bL^*) x_t \ngeqq \mathbf{0}$. However, by (D1d), $(I - bL^*) x_t + \delta_t \geqq \mathbf{0}$ must hold, which implies that $\delta_t \geq \mathbf{0}$. Then, $A^* x_t + \delta_t \geq \omega_{t-1}$, which contradicts (D1b). Therefore, $A^* x_t \leq \omega_{t-1}$ should hold, and $\delta_t = \omega_{t-1} - A^* x_t$. Thus, by (D1d), $(I - bL^*) x_t \geqq A^* x_t - \omega_{t-1}$.

((1) $\Rightarrow$ (2)) We show that at any CE in which (A^*, L^*) is adopted it must be $\pi_t^{\max} = 0$. Suppose, ad absurdum, that there exists a CE $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ with $\pi_t^{\max} > 0$. Then, by Lemma 1 and Lemma 2, $A^* x_t = \omega_{t-1}$. Then, as $(A^*, L^*) \notin \mathcal{B}_t (\omega_{t-1}, b)$, it implies that $(I - bL^*) x_t \ngeqq \mathbf{0}$. Then, as shown in the first part of the proof of Theorem 7, we derive a contradiction from (D1b)-(D1d).

((3) $\Rightarrow$ (1)) Suppose there exist $x > \mathbf{0}$ such that $(I - bL^*) x \geqq A^* x - \omega_{t-1}$ with $A^* x \leq \omega_{t-1}$ and $L^* x = N_t$; and $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$. Then, (A^*, L^*) is optimal at $(p_t, w_t) \equiv \left(\frac{1}{v^* b} v^*, \frac{1}{v^* b}\right)$. Further, let $\delta \equiv \omega_{t-1} - A^* x \geq \mathbf{0}$. Then, for all $\nu \in \mathcal{N}_t$, let

$$
\begin{aligned}
x_t^\nu &= \frac{p_t \omega_{t-1}^\nu}{p_t \omega_{t-1}} x \geqq \mathbf{0}, \;\; \delta_t^\nu = \frac{p_t \omega_{t-1}^\nu}{p_t \omega_{t-1}} \delta \geqq \mathbf{0}, \;\; l_t^\nu = 1, \\
\omega_t^\nu &= \frac{(p_t - w_t L^*) x_t^\nu + w_t - p_t b + p_t \delta_t^\nu}{p_t \left[(I - bL^*) x + \omega_{t-1} - A^* x\right]} \left[(I - bL^*) x + \omega_{t-1} - A^* x\right] \geqq \mathbf{0}.
\end{aligned}
$$

It is immediate to prove that $((A^*, L^*), (x_t^\nu, 1, \delta_t^\nu); \omega_t^\nu)$ solves MP_t^ν for all $\nu \in \mathcal{N}_t$. Furthermore, by construction, (D1b)-(D1e) are satisfied and $\pi_t^{\max} = 0$ holds. ■

2 Wages and profits when the new technique is adopted

The results in sections 5 and 6 of the paper explicitly tackle the issue of the existence of equilibrium and characterise the distribution of income at a new equilibrium induced by technical change. Thus, they hold under specific assumptions concerning, for example, technology and endowments. In this section, we relax these assumptions and analyse the effects of innovations on income distribution, under the assumption that the economy moves to a new equilibrium in which the new technique is actually adopted.[1]

Theorem A1 analyses the distributive effect of technical change in an economy which, lacking any innovations, has settled onto a steady state growth path with full employment of labour ($w_{t-1} > p_{t-1}b$) and capital ($\pi_{t-1}^{\max} > 0$).

Theorem A1: Suppose the economy is at a PCE in period $t-1$ with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$. Let $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ be profitable and labour inelastic. Suppose $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then:

(i) if (A^*, L^*) is CU-LS, then $w_t = p_t b$ and $\pi_t^{\max} > \pi_{t-1}^{\max}$ whenever $A^* x_t = \omega_{t-1}$; otherwise, $\pi_t^{\max} = 0$;

(ii) if (A^*, L^*) is CS-LU, then $\pi_t^{\max} = 0$ and the change of technique cannot be regressive.

Proof: As $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^\nu; \omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ is a PCE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$, Proposition 1 implies that $N_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1}$, and there exist $\left(\hat{\xi}_t^\nu\right)_{\nu \in \mathcal{N}_t} = (\hat{x}_t^\nu; 1; \mathbf{0})_{\nu \in \mathcal{N}_t}$ and $(\hat{\omega}_t^\nu)_{\nu \in \mathcal{N}_t}$ such that $\hat{x}_t > \mathbf{0}$ with $A_{t-1}\hat{x}_t = \omega_{t-1}$, and $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \hat{\xi}_t^\nu; \hat{\omega}_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.

Part (i). By Proposition 8 in Roemer [4], if (A^*, L^*) is profitable and CU-LS then $v^* < v_{t-1}$.

Suppose first that $A^* x_t \leq \omega_{t-1}$ holds. By Lemma 2, $p_t > \mathbf{0}$, and therefore $p_t A^* x_t < p_t \omega_{t-1}$. Then, by Lemma 1, $\pi_t^{\max} = 0$ holds.

Next, suppose that $A^* x_t = \omega_{t-1}$ holds. Because (A^*, L^*) is CU-LS, $A^* \hat{x}_t \geq A_{t-1}\hat{x}_t = \omega_{t-1}$ and $L^* \hat{x}_t < L_{t-1}\hat{x}_t = N_t$. Therefore, since $A^* x_t = \omega_{t-1}$, we obtain $A^*(\hat{x}_t - x_t) \geq \mathbf{0}$. We consider two cases.

Case 1: $0 < x_t \leqq \hat{x}_t$.

[1] In Theorems A1 and A2, we focus on PCEs such that at the beginning of t, if the status quo technique (A_{t-1}, L_{t-1}) was adopted, then $A_{t-1}^{-1}\omega_{t-1} > \mathbf{0}$ and $N_t \geqq L_{t-1}A_{t-1}^{-1}\omega_{t-1}$ would hold. If $A_{t-1}^{-1}(\omega_{t-1} - \delta) > \mathbf{0}$ and $N_t = L_{t-1}A_{t-1}^{-1}(\omega_{t-1} - \delta)$ for some $\delta \geq \mathbf{0}$, then by Proposition 1(iii) the effect of innovations on our primary variable of interest, the equilibrium profit rate, is not particularly interesting.

Clearly, $L^* x_t < L_{t-1}\hat{x}_t = N_t$, so that $w_t = p_t b$ follows from Lemma 1 and (D1c). As $p_t b = 1 = p_{t-1} b < w_{t-1}$, it follows from Theorem 2 that $\pi_t^{\max} > \pi_{t-1}^{\max}$.

Case 2: $x_t \nleqq \hat{x}_t$.

We only need to show $L^* x_t < L_{t-1}\hat{x}_t$. The rest of the proof then follows as in case 1. Suppose, by way of contradiction, that $L^* x_t \geqq L_{t-1}\hat{x}_t = N_t$. By (D1c), this implies $L^* x_t = N_t$. Given $L^* \leq L_{t-1}$ and $x_t > \mathbf{0}$, this implies $L_{t-1} x_t > L^* x_t = N_t$. Next, $A^* x_t = \omega_{t-1}$, and $A_{t-1}\hat{x}_t = \omega_{t-1}$ imply $N_t = L_{t-1} A_{t-1}^{-1} A_{t-1} \hat{x}_t = L_{t-1} A_{t-1}^{-1} A^* x_t$. Therefore $L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t < 0$.

Because (A^*, L^*) is labour inelastic, it follows that

$$(L^* - L_{t-1}) x_t < L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t = L_{t-1} A_{t-1}^{-1} A^* x_t - L_{t-1} x_t,$$

which implies $L^* x_t < L_{t-1} A_{t-1}^{-1} A^* x_t = N_t$, which yields the desired contradiction.

Part (ii). Suppose, ad absurdum, that $\pi_t^{\max} > 0$. By Lemma 1, $p_t A^* x_t^{\nu} = p_t \omega_{t-1}^{\nu}$, all $\nu \in \mathcal{N}_t$ and by Lemma 2, $p_t > \mathbf{0}$. Therefore by (D1b), $A^* x_t = \omega_{t-1}$ and, noting that $L_{t-1} x_t = L_{t-1} A_{t-1}^{-1} \omega_{t-1} = N_t$ it follows that $L_{t-1} A_{t-1}^{-1} A^* x_t = N_t$. By (D1c), and noting that $L^* \geq L_{t-1}$ and $x_t > \mathbf{0}$, it follows that $N_t \geqq L^* x_t > L_{t-1} x_t$. Therefore $L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t > 0$. Because (A^*, L^*) is labour inelastic, $(L^* - L_{t-1}) x_t > L_{t-1} A_{t-1}^{-1} (A^* - A_{t-1}) x_t = N_t - L_{t-1} x_t$, which implies $L^* x_t > N_t$, in contradiction with (D1c).

To see that (A^*, L^*) cannot be regressive, observe that if $\pi_t^{\max} = 0$ at the CE, then $(p_t, w_t) = \left(\frac{v^*}{v^* b}, \frac{1}{v^* b}\right)$. As (A^*, L^*) is optimal at prices (p_t, w_t), it follows that $v^* \leqq v^* A_{t-1} + L_{t-1}$. Thus, $v^* \leqq v_{t-1}$ holds, and technical change cannot be regressive. ■

Suppose the economy is on a growth path with full employment of productive factors, but a new technique (A^*, L^*) emerges, at the end of period $t-1$, and it is profitable. If (A^*, L^*) is adopted in equilibrium, then by Theorem A2(i) two things can happen: if (A^*, L^*) is CU-LS, and it leads to the emergence of an excess supply of labour and unemployment, then the profit rate increases and the wage rate falls to the subsistence level. This is the Marxian "industrial reserve army of the unemployed". Together, Theorem 1, Proposition 1, and Theorem A2 may be interpreted as illustrating Marx's [3] general law of capitalist accumulation. If, however, (A^*, L^*) is CS-LU, or more generally the shift to the new technique makes aggregate capital abundant relative to the labour force, then the equilibrium profit rate falls to zero.[2]

Theorem A2 characterises equilibria with a new technique when the aggregate capital stock at $t-1$ is *not* sufficient to allow for the full employment of labour using the status quo technique ($A_{t-1}x_{t-1} = \omega_{t-2}$ and $L_{t-1}x_{t-1} < N_{t-1}$):

Theorem A2: Suppose the economy is at a PCE in period $t-1$ with $A_{t-1}x_{t-1} = \omega_{t-2}$ and $L_{t-1}x_{t-1} < N_{t-1}$. Let $(A^*, L^*) \in \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ be profitable and labour inelastic. Suppose $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. If (A^*, L^*) is either CU-LS or CS-LU with sufficiently small $(A_{t-1} - A^*, L^* - L_{t-1})$, then $w_t = p_t b$ and $\pi_t^{\max} > \pi_{t-1}^{\max}$ whenever $A^* x_t = \omega_{t-1}$; otherwise, $\pi_t^{\max} = 0$.

Proof: 1. As the CE in period $t-1$ is persistent, Proposition 1 implies that $N_t \geqq L_{t-1}A_{t-1}^{-1}\omega_{t-1}$, and there exist $\left(\hat{\xi}_t^\nu; \hat{\omega}_t^\nu\right)_{\nu \in \mathcal{N}_t}$ such that $\hat{x}_t > \mathbf{0}$ with $A_{t-1}\hat{x}_t = \omega_{t-1}$ and $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \hat{\xi}_t^\nu; \hat{\omega}_t^\nu\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.
2. If $A^* x_t \leq \omega_{t-1} = A_{t-1}\hat{x}_t$, then the result follows as in the proof of Theorem A1(i). Therefore consider the case with $A^* x_t = \omega_{t-1}$.
3. Let (A^*, L^*) be CU-LS. Suppose $L^* x_t \geqq L_{t-1}\hat{x}_t$. Because $L^* \leq L_{t-1}$ and $x_t > \mathbf{0}$, it follows that $L_{t-1}x_t > L^* x_t$. Noting that $L_{t-1}\hat{x}_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1}$, we obtain $L_{t-1}A_{t-1}^{-1}(A^* - A_{t-1})x_t < 0$. Since (A^*, L^*) is labour inelastic, it follows that $(L^* - L_{t-1})x_t < L_{t-1}A_{t-1}^{-1}(A^* - A_{t-1})x_t < 0$, which in turn implies $L^* x_t < L_{t-1}x_t$, yielding the desired contradiction. Thus, $L^* x_t < L_{t-1}\hat{x}_t \leqq N_t$. Therefore, Theorem 1(ii) implies $w_t = p_t b$, which in turn implies $\pi_t^{\max} > \pi_{t-1}^{\max}$ by Theorem 2.
4. Let (A^*, L^*) be CS-LU. Suppose that $N_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1}$. As $A^* x_t = \omega_{t-1}$, $N_t = L_{t-1}A_{t-1}^{-1}A^* x_t$ holds. As in the proof of Theorem A1(ii), it can be shown that $N_t \geqq L^* x_t > L_{t-1}x_t$ holds, and so $L_{t-1}A_{t-1}^{-1}(A^* - A_{t-1})x_t > 0$. Then, as in the proof of Theorem A1(ii), the labour inelasticity of (A^*, L^*) implies that $L^* x_t > N_t$, in contradiction with (D1c). Therefore, given that $N_t \geqq L_{t-1}A_{t-1}^{-1}\omega_{t-1}$ holds, we cannot but conclude that $L_{t-1}A_{t-1}^{-1}\omega_{t-1} < N_t$. Since $(A_{t-1} - A^*, L^* - L_{t-1})$ is sufficiently small, $L^* A^{*-1}\omega_{t-1}$ is sufficiently

[2] If (A^*, L^*) is CS-LU and regressive, then it will not be adopted in equilibrium consistent with Theorems 5 and 6.

close to $L_{t-1}A_{t-1}^{-1}\omega_{t-1}$, which implies that $N_t > L^* A^{*-1}\omega_{t-1}$ holds, and thus $N_t > L^* x_t$ under $A^* x_t = \omega_{t-1}$. Then, $w_t = p_t b$ follows from Theorem 1(ii), and by Theorem 2, $\pi_t^{\max} > \pi_{t-1}^{\max}$. ■

3 The falling rate of profit and capital theory

As mentioned in section 6 of the paper, Theorem 5 shows some interesting and perhaps surprising connections between the theory of the falling profit rate and some central insights of classical capital theory.

As an illustration, and without any loss of generality, consider the simplest possible case of technical change, whereby only one technique is known in period $t-1$, so that $\mathcal{B}_{t-1} = \{(A, L)\}$ and $\mathcal{B}_t = \{(A, L), (A^*, L^*)\}$. Under the conditions of Theorem 5, the wage-profit curve of the new technique, $\pi w(A^*, L^*)$, dominates the wage-profit curve of (A, L), $\pi w(A, L)$, at least in a neighbourhood of points $\left(0, \frac{1}{v^* b}\right)$ and $(\Pi(A^*), 0)$,[3] as well as in the non-empty subset $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$.[4]

Then, there are two scenarios in which the profit rate will fall. In the first, $\pi w(A^*, L^*)$ completely dominates $\pi w(A, L)$ as shown in Figure A1.

Insert Figure A1 around here.

In this case, technical change is profitable at *any* prices and yet, according to Theorem 5 the adoption of (A^*, L^*) leads the equilibrium profit rate to drop to zero. This is quite a strong – and perhaps surprising – result from a theoretical viewpoint, but it is possibly of limited empirical relevance, because innovations that are profitable at any prices are rare.

Alternatively, if $\pi w(A^*, L^*)$ does not completely dominate $\pi w(A, L)$, and given that the former dominates the latter in at least three regions, the two curves must intersect at least twice, as shown in Figure A2.

Insert Figure A2 around here.

[3]The former follows noting that if the condition in Theorem 5 holds, then $\frac{1}{v'b} \leqq \frac{1}{v^* b}$ for all $(A', L') \in \mathcal{B}_t$ and $\pi w(A^*, L^*)$ coincides with the wage-profit frontier $\pi w(\mathcal{B}_t)$ in a neighbourhood of $(\pi_t^{\max}, w_t) = \left(0, \frac{1}{v^* b}\right)$. The latter follows noting that $A^* \leq A$ implies $\Pi(A^*) > \Pi(A)$.

[4]Because technical change is profitable, an argument similar to that used for Theorem 3 shows that the set $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$ is non-empty and coincides with $\pi w(\mathcal{B}_t)$.

Figure A2 describes a situation in which a *reswitching of techniques* (Kurz and Salvadori [2], p.148) occurs: because $\frac{1}{vb} < \frac{1}{v^*b}$, close to the vertical axis the wage-profit frontier coincides with the wage-profit curve of the technique (A^*, L^*), which is therefore optimal for a sufficiently small (or zero) profit rate. Further, as (A^*, L^*) is the optimal technique at $\pi^* = 0$, the corresponding wage rate, $w^* = \frac{1}{v^*b}$ is higher than the wage rate, $w = \frac{1}{vb}$, associated with $\pi = 0$ under (A, L). In this case, as well-known in the literature on the Cambridge capital controversy, the capital-labour ratio of (A^*, L^*) is higher than that of (A, L) when the values of capital are measured by means of the commodity price vectors corresponding to each of the two switching points, and so (A^*, L^*) is a more capital-intensive technique than (A, L).

As the profit rate increases, a switching point arrives after which the frontier coincides with $\pi w(A, L)$ and the more labour-intensive technique (A, L) becomes optimal. However, since $\Pi(A^*) > \Pi(A)$ another switching point exists after which, as the profit rate *increases* further, the *capital intensive* technique (A^*, L^*) becomes optimal again – a phenomenon known in the literature as *capital reversing* (Kurz and Salvadori [2], pp.447-451).

In other words, setting aside the empirically less relevant case of an innovation unambiguously dominating older techniques, the above arguments show that there exists an interesting relation between capital theory – and the phenomena known as reswitching of techniques and capital reversing, – and the theory of the falling profit rate.

4 CS-LU Technical Change in Developing Economies

Consider a developing economy in which the social endowments of capital stocks accumulated in the past are still very low relative to the size of population. In this case, it is natural to assume that a PCE is characterised by $A_{t-1}x_{t-1} = \omega_{t-2}$ and $L_{t-1}x_{t-1} < N_{t-1}$ and ask whether the premise of Theorem A2 can be satisfied. This is particularly relevant if a CS-LU change of technique is considered, as in the next result.

Theorem A3: Suppose the economy is at a PCE in period $t-1$ with $A_{t-1}x_{t-1} = \omega_{t-2}$ and $L_{t-1}x_{t-1} < N_{t-1}$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be profitable, labour inelastic, and CS-LU with sufficiently small $(A_{t-1} - A^*, L^* - L_{t-1})$.

Then, there exists a CE $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ with $w_t = 1$ and $\pi_t^{\max} > \pi_{t-1}^{\max}$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

Proof: Following the proof of Theorem A2, we can see that $N_t > L^* A^{*-1} \omega_{t-1}$ holds. Then, let

$$\begin{aligned} x_t^\nu &= \frac{p_t \omega_{t-1}^\nu}{p_t \omega_{t-1}} A^{*-1} \omega_{t-1}, \ \ l_t^\nu = \frac{L^* A^{*-1} \omega_{t-1}}{N_t}, \ \ \delta_t^\nu = \mathbf{0}, \\ \omega_t^\nu &= \frac{p_t x_t^\nu - w_t L^* x_t^\nu + (w_t - 1) l_t^\nu}{(p_t - L^*) A^{*-1} \omega_{t-1}} (I - bL^*) A^{*-1} \omega_{t-1}. \end{aligned}$$

Noting that $A_{t-1} x_{t-1} = \omega_{t-2}$ and $L_{t-1} x_{t-1} < N_{t-1}$ imply $w_{t-1} = 1$, it follows that $((A^*, L^*); \xi_t^\nu; \omega_t^\nu)$ solves MP_t^ν for all $\nu \in \mathcal{N}_t$ and (D1b)-(D1e) are satisfied for $(\pi_t^{\max}, w_t = 1) \in \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$ with $p_t = w_t L^* \left[I - (1 + \pi_t^{\max}) A^*\right]^{-1} > \mathbf{0}$, as in the proof of Theorem 4. Thus, $\left((p_t, w_t), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ constitutes a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. By Theorem A2, $\pi_t^{\max} > \pi_{t-1}^{\max}$ holds. ■

Theorem A3 shows that the premise of Theorem A2 is satisfied: if there is a sufficiently big industrial reserve army of the unemployed, then a profitable, gradual, CS-LU change of technique will indeed be adopted in equilibrium, and lead to an increase in the profit rate, even if this change of technique is regressive.

Both the assumption $A_{t-1} x_{t-1} = \omega_{t-2}$ and $L_{t-1} x_{t-1} < N_{t-1}$, and the characteristics of the new equilibrium described in Theorem A3 are quite realistic in developing economies, in which aggregate labour is abundant relative to the level of accumulated capital stock. These economies may wish to import the advanced technology (a more capital-intensive technique) from advanced economies, but their aggregate capital endowments are often insufficient to adopt capital-intensive techniques. In this case, developing economies may modify such advanced technology into a slightly more labour-intensive one, as in the case of the Japanese economy just after the Meiji Revolution around the mid 19th century (see, e.g., Allen [1]).

References

[1] Allen, J., 2011. *Global Economic History: A Very Short Introduction.* Oxford University Press, Oxford.

[2] Kurz, D.K., Salvadori, N., 1995. *Theory of Production: A Long-Period Analysis*. Cambridge University Press, Cambridge.

[3] Marx, K., 1954 [1867]. *Capital*, Vol.I. Lawrence & Wishart, London.

[4] Roemer, J.E., 1977. Technical change and the 'tendency of the rate of profit to fall'. *Journal of Economic Theory* 16, 403-424.

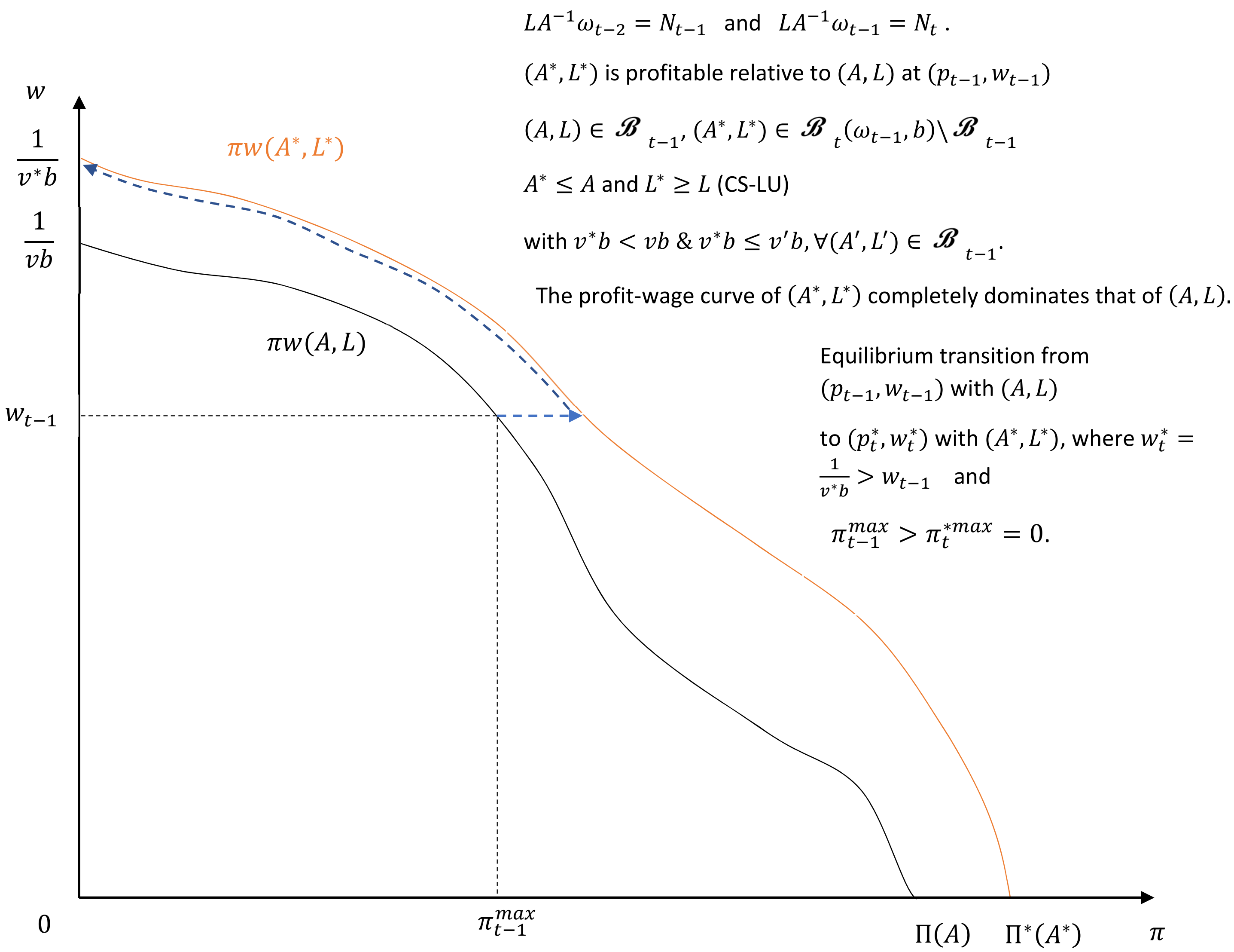
$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t$.
(A^*, L^*) is profitable relative to (A, L) at (p_{t-1}, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}, (A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU)
with $v^*b < vb$ & $v^*b \leq v'b, \forall (A', L') \in \mathscr{B}_{t-1}$.
The profit-wage curve of (A^*, L^*) completely dominates that of (A, L).
Equilibrium transition from (p_{t-1}, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A^*, L^*), where $w_t^* = \frac{1}{v^*b} > w_{t-1}$ and
$\pi_{t-1}^{max} > \pi_t^{*max} = 0$.
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
$\pi w(A^*, L^*)$
$\pi w(A, L)$
w_{t-1}
0
π_{t-1}^{max}
$\Pi(A)$
$\Pi^*(A^*)$
π

Figure A1

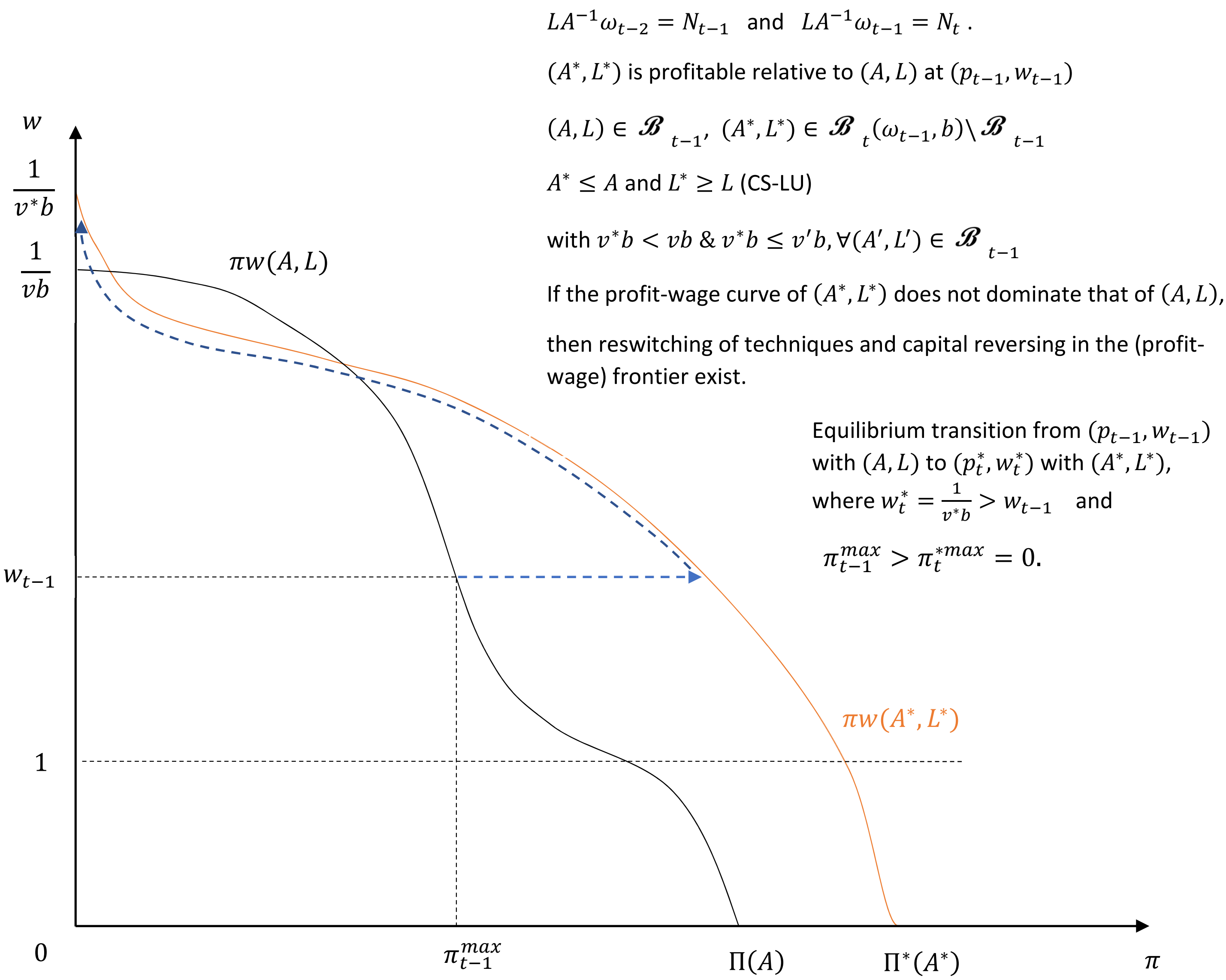
$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t$.
(A^*, L^*) is profitable relative to (A, L) at (p_{t-1}, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}$, $(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \setminus \mathscr{B}_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU)
with $v^*b < vb$ & $v^*b \leq v'b, \forall (A', L') \in \mathscr{B}_{t-1}$
If the profit-wage curve of (A^*, L^*) does not dominate that of (A, L),
then reswitching of techniques and capital reversing in the (profit-wage) frontier exist.
Equilibrium transition from (p_{t-1}, w_{t-1}) with (A, L) to (p_t^*, w_t^*) with (A^*, L^*),
where $w_t^* = \frac{1}{v^*b} > w_{t-1}$ and
$\pi_{t-1}^{max} > \pi_t^{*max} = 0.$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
$\pi w(A, L)$
w_{t-1}
$\pi w(A^*, L^*)$
1
0
π_{t-1}^{max}
$\Pi(A)$
$\Pi^*(A^*)$
π

Figure A2