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Abstract

We study pillage games (Jordan in J Econ Theory 131.1:26–44, 2006, "Pillage and property"), which model unstructured power contests. To enable empirical tests of pillage game theory, we relax a symmetry assumption that agents' intrinsic contributions to a coalition's power is identical. We characterise the core for all n. In the three-agent game: (i) only eight configurations are possible for the core, which contains at most six allocations; (ii) for each core configuration, the stable set is either unique or fails to exist; (iii) the linear power function creates a tension between a stable set's existence and the interiority of its allocations, so that only special cases contain strictly interior allocations. Our analysis suggests that non-linear power functions may offer better empirical tests of pillage game theory.

Key words: power contests, core, stable sets

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1 Introduction

We study allocation of resources in unstructured power contests. The underlying model is that of a *pillage game* (Jordan, 2006) in which an aggregate wealth

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endowment is split among agents, who can form coalitions. A *power function* maps every coalition and its wealth to a real number. A more powerful coalition can non-consensually 'pillage' the resources of a less powerful coalition. As a pillage game's power function completely encodes its dominance relation, this corresponds to the new allocation dominating the original one. The power function is assumed to be monotonic, so that coalitions become more powerful as they gain members, or gain resources.

A typical game-theoretic solution describes allocations of wealth (analogous to imputations in more general coalitional games) corresponding to the core (which may be empty), or a von Neumann-Morgenstern stable set (which may not exist). The core is appealing as the set of undominated allocations. As farsighted agents may recognise that a pillage operation that benefits them may allow another that does not (Harsanyi, 1974), Jordan (2006) proved that, when agents shared common expectation functions over dominance paths, stable sets are equivalent to farsighted cores.[1]

The existing literature on pillage games almost exclusively considers *symmetric* power functions, in which the identity of the agents in a coalition does not matter. We consider the simplest generalisation of this model: each agent has an intrinsic 'strength' that is added to its wealth endowment; the power of a coalition is the sum of the wealth and the intrinsic strengthes of its members. We term such games *asymmetric majority pillage games* (AMPG); they are parameterised by a number of players, and a vector of intrinsic strengths.

In the classic majority game (in characteristic function form), one allocation dominates another if and only if it is preferred by a strictly larger coalition. In the majority pillage game (Jordan and Obadia, 2015), an allocation may also dominate another if the coalitions favouring and opposing it are the same size, as long as the former holds more resources. The AMPG studied here generalise the majority pillage game, both by allowing asymmetric intrinsic strength and by allowing the non-empty core case (in which resource holdings can dominate intrinsic strength).

As symmetry is the exception in practice, we relax it to aid both applications and tests of the theory. We are particularly interested in when stable sets exist, and when they allow interior allocations, so that no agent is left without resources.

We believe that pillage games are of interest for three reasons. First, we see unstructured interactions, without binding game forms, as pervasive: "procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak" (Aumann, 1985). Second, we believe that resources often convey both utility and power. Third, we see the costless, predictable nature of pillage as consistent with

[1]This result has contributed to the deepest body of resulting theory, q.v. Ray and Vohra (2015), Dutta and Vohra (2017), Ray and Vohra (2019), Bloch and Nouweland (2020).

sophisticated, well-informed agents, even in the presence of strong institutions: for example, by convention, losers of democratic elections publicly concede, signalling to their supporters not to engage in costly civil wars that they will lose.[2]

Pillage games' stable sets can yield sharp predictions: they are small (Jordan, 2006; Kerber and Rowat, 2011; Saxton, 2011; Rowat and Kerber, 2014; Beardon and Rowat, 2013) and unicity seems to be the norm (Jordan, 2006; Rowat and Kerber, 2014).[3]

Section 2 introduces pillage games. Theorem 1 establishes that only two types of allocations may belong to the core of AMPGs for any $n \geq 3$, and the conditions under which they do. For the $n = 3$ case, Corollary 1 identifies the corresponding strength vector for each of the eight possible cores.

The rest of the paper focuses on the three-agent AMPG. Thorough $n = 3$ analyses have typically been important steps towards developing more general analyses in the coalitional literature.[4] This is partly pragmatic:

> Finding stable sets involves a new tour de force of mathematical reasoning for each game or class of games that is considered. ... there is no theory, no tools, certainly no algorithm. (Aumann, 1985)

Of course, $n = 3$ is not just a useful stepping stone, but an appropriate model of some situations — from the geopolitical (e.g. the Chinese Three Kingdoms period of 220 – 280 AD) to the microeconomic (e.g. (Straffin, 1993; McDonald, 1977) on sharing communications satellites, or Hellman and Wasserman (2011) on splitting founders' equity).

Section 3 analyses stable sets when the core is empty. Theorem 2 presents necessary and sufficient conditions for the existence of a unique stable set comprising three allocations, each splitting the resource equally between two of the agents. Perhaps surprisingly, this is identical to the stable set in the symmetric special case: power struggles over these 50/50 allocations pit one agent (without resources) against another agent (with half the resources); as long as the former's

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[2] Jackson and Morelli (2011) open their survey of the reasons for wars by asking why they, "occur and recur, especially in cases when the decisions involved are made by careful and rational actors?" Pillage games aid study of the cases short of war.

[3] MacKenzie, Kerber, and Rowat (2015) constructed examples of pillage games with multiple stable sets. It is also the only paper to violate symmetry, which it did to construct counterexamples, rather than systematically.

[4] For instance, von Neumann and Morgenstern (1953) for games in characteristic function form, and Thrall and Lucas (1963) for those in partition function form. These both establish results in the simplest non-trivial case, and build intuitions for further analyses. This partly reflects the computational complexity of the stable set as a solution concept: even in a special case, Deng and Papadimitriou (1994) found that "existence [of a stable set] may not even be decidable"

intrinsic strength is not too much greater than the latter's, the 50/50 split is defensible; otherwise, the stable set fails to exist. Thus, power asymmetries do not distort the stable set: they strain it to the point at which it breaks.

Section 4 analyses stable sets for non-empty cores. Although the geometry is more complicated, we may proceed algorithmically: first, if a stable set exists, the core must belong to it; second, allocations dominated by the core cannot; third, the remaining allocations — typically linear loci within which one agent is as powerful as the other two — can be analysed using techniques from the empty core case. Corollary 4 presents a general non-existence result; the remainder of the section presents results for stable sets under each of the eight possible cores. It finds a tension between a stable set's existence and the possibility of interior allocations: as candidate interior allocations balance agents' powers, they cannot dominate their neighbouring allocations; domination of the other allocations on the locus typically requires including an extremal allocation, which lies on the simplex's edge; when agents have positive intrinsic strength, these extremal allocations are themselves dominated by core allocations, preventing existence.

Section 5 concludes the paper. Its overarching findings are: non-existence of stable sets is pervasive in $n = 3$ AMPG; strictly interior allocations only occur in special cases — including both new configurations and the symmetric case already known to the literature (Jordan, 2006, Theorem 3.3); when stable sets do exist, they are unique. The link between the linear power function used here and the non-existence mechanism suggests that non-linear power functions may by better candidates for empirical work.

2 Pillage games

Let $I \equiv \{1, \ldots, n\}$ be a finite set of *agents*, indexed by i. An *allocation*, $\boldsymbol{x}$, is a division of a unit resource among them, so that the *feasible set of allocations* is

$$X \equiv \left\{ (x_i)_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}. \tag{1}$$

Let $\subset$ denote a proper set inclusion, and use $\subseteq$ to allow the possibility of equality. Jordan (2006) defined a *power function* over subsets of agents and allocations, so that $\pi : 2^I \times X \to \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$, then $\pi(C', \boldsymbol{x}) \geq \pi(C, \boldsymbol{x})$ for all $\boldsymbol{x} \in X$;

(WR) if $y_i \geq x_i$ for all $i \in C \subseteq I$, then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

(SR) if $\varnothing \neq C \subseteq I$ and $y_i > x_i$ for all $i \in C$, then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Simply, the axioms imply that more is better: by axiom WC, adding agents makes a coalition more powerful; by axiom WR, weakly adding resources makes a coalition weakly more powerful; axiom SR is a strict version of axiom WR.

The class of power functions considered here is

$$\pi(C, \boldsymbol{x}) = \sum_{i \in C} (x_i + v_i); \tag{2}$$

where $C \subseteq I$ is a non-empty coalition of agents, $\boldsymbol{x} \in X$ and $v_i \in \mathbb{R}_+$ allows agents' intrinsic strengths to differ. Power function (2) provides one of the simplest possible specifications of an asymmetric pillage game, introducing one parameter per agent. Jordan's *wealth-is-power* function is a symmetric special case of this in which $v_i = 0$ for all i; the *majority pillage game* analysed in Jordan and Obadia (2015) sets $v_i > 1$ for all i.

We index agents in two different ways. First, to merely label them, we use the generic $i, j, k, \ldots$ or $1, 2, 3, \ldots$. Second, to order agents by their intrinsic contributions to their coalitions' power, we use a, b and c such that

$$v_a \geq v_b \geq \cdots \geq v_n \geq 0. \tag{3}$$

If we display an allocation's constituent coordinates, we do so in the natural order implied by our choice of index, whether $\boldsymbol{x} = (x_1, x_2, x_3)$ or $\boldsymbol{x} = (x_a, x_b, x_c)$. This does not imply that, for example, $x_a = x_1$.

An allocation $\boldsymbol{y}$ *dominates* an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, iff

$$\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x});$$

where $W \equiv \{i \,|y_i > x_i\}$ and $L \equiv \{i \,|x_i > y_i\}$ are called the *win set* and *lose set*, respectively. Thus, allocation $\boldsymbol{y}$ dominates allocation $\boldsymbol{x}$ if and only if the set of agents who benefit in a re-allocation from $\boldsymbol{x}$ to $\boldsymbol{y}$ are more powerful at the original allocation $\boldsymbol{x}$ than is the set of agents who lose from that re-allocation.[5]

By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric; as in the general case (von Neumann and Morgenstern, 1953), dominance is not generally transitive.

For $Y \subset X$, let

$$D(Y) \equiv \{\boldsymbol{x} \in X \,|\exists \boldsymbol{y} \in Y \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\} \tag{4}$$

be the *dominion* of Y, the set of allocations dominated by an allocation in Y.

This paper studies asymmetric majority pillage games:

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[5]When referring to two allocations in the following, we may use W and L as a shorthand to indicate the agents benefiting and losing, respectively, from a move between them, even if we do not explicitly define them as such.

Definition 1. *An* asymmetric majority pillage game *(AMPG) is a profile* $\langle I, X, (v_i)_{i\in I}, \pi\rangle$*, where* $I = \{1, 2, \ldots, n\}$*, X is defined by equation* (1)*,* $(v_i)_{i\in I}$ *are non-negative reals, and the power function* π *is defined by* (2)*.*

For tractability's sake, we mainly consider the $n = 3$ case. Let 3-AMPG denote an *AMPG* with $n = 3$.

2.1 The core

The *core* is the set of undominated allocations, $K \equiv X \backslash D(X)$. To characterise it, define two types of allocation:

Definition 2. *(Jordan, 2006) Let* $\boldsymbol{t}^i \in X$ *be a* tyrannical *allocation such that* $t^i_i = 1$ *and* $t^i_j = 0$ *for all* $j \neq i \in I$*.*

Let $\boldsymbol{b}^{ij} \in X$ *be a* bilaterally balanced *allocation, so that* $\pi\left(\{i\}, \boldsymbol{b}^{ij}\right) = \pi\left(\{j\}, \boldsymbol{b}^{ij}\right)$ *with* $b^{ij}_i, b^{ij}_j > 0$ *for distinct i, j and* $b^{ij}_k = 0$ *for any other agents k.*

Note that a tyrannical allocation may be a limit point of a sequence of bilaterally balanced allocations, but an allocation cannot be both tyrannical and bilaterally balanced.

Thus, given power function (2), we have:

$$b^{ij}_i = \tfrac{1}{2}\left(1 - v_i + v_j\right). \tag{5}$$

Then:

Theorem 1. *In the AMPG with* $n \geq 3$*, the core can contain no allocations other than the tyrannical* $\boldsymbol{t}^i$ *and the bilaterally balanced* $\boldsymbol{b}^{ij}$*, for all distinct i and j:*

1. $\boldsymbol{t}^i$ *belongs to the core iff* $v_i \geq \sum_{j\in I\backslash\{i\}} v_j - 1$*; and*
2. $\boldsymbol{b}^{ij}$ *belongs to the core iff* $v_k = 0$ *for all k distinct from i, j, and* $v_i - v_j \in (-1, 1)$*.*

Proof. First, consider the allocations that assign the resource exclusively to a single agent. These are, by definition, the tyrannical $\boldsymbol{t}^i$. The theorem's first condition is equivalent to the non-existence of $\boldsymbol{x} \in X\backslash\left\{\boldsymbol{t}^i\right\}$ such that $\boldsymbol{x} \succ \boldsymbol{t}^i$.

Now consider the allocations that split the resource between two agents, say i and j, so that $x_k = 0$ for all other agents k. For these agents to defend their holdings against the other two, the following inequalities must be satisfied:

$$x_i + v_i \geq x_j + v_j + \sum_{k\in I\backslash\{i,j\}} v_k;$$
$$x_j + v_j \geq x_i + v_i + \sum_{k\in I\backslash\{i,j\}} v_k.$$

Combining these requires

$$x_j + v_j + 2 \sum_{k \in I \setminus \{i,j\}} v_k \leq x_i + v_i + \sum_{k \in I \setminus \{i,j\}} v_k \leq x_j + v_j.$$

As x_i, x_j, v_i and v_j are non-negative, this forces $\sum_{k \in I \setminus \{j,k\}} v_k = 0$, forcing all individual $v_k = 0$. Thus, $x_i + v_i = x_j + v_j$ which, with the unit endowment constraint, yields the $\boldsymbol{b}^{ij}$. Constraining b_i^{ij} and b_j^{ij} to lie in $(0, 1)$ yields the interval condition for v_j in the statement of the theorem.

Finally consider allocations that split the resource between three agents, so that $x_i, x_j, x_k > 0$. By the above reasoning, one of the necessary conditions is

$$x_j + v_j + 2x_k + 2v_k + 2 \sum_{l \notin \{i,j,k\}} v_l \leq x_i + v_i + x_k + v_k + \sum_{l \notin \{i,j,k\}} v_l \leq x_j + v_j.$$

This requires $x_k = 0$, a contradiction. The possibility of core allocations with more than three agents holding resources is similarly ruled out. □

As a $\boldsymbol{b}^{ij}$ can belong to the core only if both $\boldsymbol{t}^i$ and $\boldsymbol{t}^j$ do, the following correspondences between cores and parameters holds for $n = 3$:

Corollary 1. *1.* $K = \varnothing$:

$$-v_a + v_b + v_c > 1. \tag{6}$$

2. $K = \{\boldsymbol{t}^a\}$:

$$-v_a + v_b + v_c \leq 1, -v_a + v_b - v_c < -1. \tag{7}$$

3. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$:

$$(-v_a + v_b - v_c \geq -1), (0 < v_c < v_a + v_b - 1);\ or \tag{8}$$

$$v_c = 0 < v_b = v_a - 1. \tag{9}$$

4. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$:

$$v_c = 0, v_b \geq v_a - 1, v_b > 1 - v_a. \tag{10}$$

5. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$:

$$v_c \geq v_a + v_b - 1, v_c > 0; or \tag{11}$$

$$v_a = 1 > v_b = v_c = 0. \tag{12}$$

6. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$:

$$v_c = 0 < v_b \leq 1 - v_a. \tag{13}$$

7. $K = \left\{ \boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac} \right\}$:

$$0 = v_c = v_b < v_a < 1. \tag{14}$$

8. $K = \left\{ \boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc} \right\}$:

$$v_c = v_b = v_a = 0. \tag{15}$$

The corollary follows from the theorem's requirement that each agent holding resources must be at least as powerful as the other two combined: the cases instantiate the theorem's conditions for the inclusion or exclusion of the $\boldsymbol{t}^i$ and $\boldsymbol{b}^{ij}$.

Empty core condition (6) thus generalises the symmetric condition in Jordan (2006, Proposition 4.2), which was restricted to $v > 1$.

2.2 Stable sets

The *stable set* is the original von Neumann and Morgenstern (1953) solution concept, initially just called the 'solution'. Unlike the core, which is defined pointwise, it is defined setwise, making it harder to compute. Intuitively, a set of allocations is stable if they satisfy internal stability (no allocation in a stable set dominates another) and external stability (every allocation outside a stable set is dominated by at least one allocation in a stable set).

A set of allocations, $S \subseteq X$, is a *stable set* iff it satisfies *internal stability*,

$$S \cap D(S) = \varnothing; \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

The conditions combine to yield $S \equiv X \backslash D(S)$.

While stable sets may not exist, or may be non-unique, the core necessarily belongs to any stable set; when the core also satisfies external stability, it is the unique stable set. Jordan (2006) proved that a pillage game's stable set has the property of being the set of allocations that are undominated given a consistent set of expectations about what subsequent domination operation would be attempted following the initial one.

The rest of this paper analyses stable sets in the 3-AMPG, seeking to decide existence and — when they do exist — to derive them.

3 The empty core

The analysis in this section extends that in Jordan and Obadia (2015, §3) to the asymmetric case. The core is empty when inequality (6) holds.

The following lemma generalises Jordan and Obadia (2015, Lemma 3.4) beyond its symmetric case of $v_1 = v_2 = v_3 > 1$:

Lemma 1. *Suppose that S is internally stable in a 3-AMPG with an empty core. If $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to S, then there exists an $i \in I$ such that $x_i = x_i'$. Further, $x_k + v_k \leq x_j + v_j$ and $x_j' + v_j \leq x_k' + v_k$.*

Proof. Assume, contrary to the lemma, that there exist $\boldsymbol{x}, \boldsymbol{x}' \in S$ such that $x_i \neq x_i'$ for all $i \in I$. Then, without loss of generality, we generically have that $x_i > x_i'$, $x_j < x_j'$ and $x_k < x_k'$. This yields a contradiction:

1. the empty core property requires that $x_i + v_i < 1 - x_i + v_j + v_k$ for all $\boldsymbol{x} \in X$ and distinct $i, j, k \in I$;

2. internal stability requires that $\boldsymbol{x}' \not\prec\!\!- \boldsymbol{x}$, so that $x_i + v_i \geq 1 - x_i + v_j + v_k$.

Thus, for two internally stable allocations there must exist at least one agent for whom $x_i = x_i'$. When these allocations are distinct, $x_j - x_j' = -\left(x_k - x_k'\right) \neq 0$, so that $W \equiv \left\{i \in I \,\middle|\, x_i > x_i'\right\}$ and $L \equiv \left\{i \in I \,\middle|\, x_i < x_i'\right\}$ are singletons that exclude i. □

Intuitively, the empty core ensures that the loci of allocations at which one agent is just as powerful as the other two lie outside the set of feasible allocations, X. In internally stable sets all pairs of allocations must lie on opposite sides of such balance of power loci. Thus, given an empty core, an internally stable set cannot contain two allocations that pit a given agent against the other two. Along the relevant balance of power loci, then, at least one agent must be indifferent.

Now extend Jordan and Obadia's Lemma 3.5 to the asymmetric case:

Lemma 2. *Suppose that S is internally stable in a 3-AMPG with an empty core. Then S has no more than three elements*

The proof establishes that, under the lemma's conditions, an internally stable set cannot contain four allocations, which also precludes it having more than four:

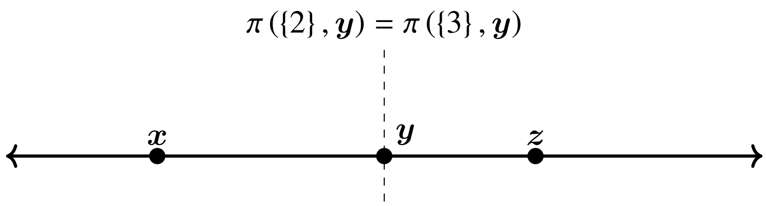

Figure 1: Collinear, internally stable $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ with $x_1 = y_1 = z_1$

Proof. Consider three points, $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{z}$, in an internally stable set S. By Lemma 1, the line running between any two of these must be parallel to the edge of the simplex X. Without loss of generality, there are two possible configurations:

1. $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$, are collinear, as depicted in Figure 1, with (wlog) $x_1 = y_1 = z_1$, $x_2 > y_2 > z_2$, and $x_3 < y_3 < z_3$.

   In this case, as $\pi(\{2\}, \boldsymbol{x}) > \pi(\{2\}, \boldsymbol{y}) > \pi(\{2\}, \boldsymbol{z})$ and $\pi(\{3\}, \boldsymbol{x}) < \pi(\{3\}, \boldsymbol{y}) < \pi(\{3\}, \boldsymbol{z})$ hold, internal stability requires that $\pi(\{2\}, \boldsymbol{y}) = \pi(\{3\}, \boldsymbol{y})$. Assume, on the contrary, there is another distinct $\boldsymbol{w} \in S$. Then there are two cases to consider:

   (a) $w_1 = x_1$. Then we have the following four possibilities, each leading to a contradiction: i. $w_2 > x_2$, so that $\boldsymbol{w} \mathrel{\varepsilon\!-} \boldsymbol{x}$ holds by $v_2 + x_2 > v_3 + x_3$; ii. $x_2 > w_2 > y_2$, so that $\boldsymbol{x} \mathrel{\varepsilon\!-} \boldsymbol{w}$; iii. $y_2 > w_2 > z_2$, so that $\boldsymbol{z} \mathrel{\varepsilon\!-} \boldsymbol{w}$; and iv. $z_2 > w_2$, so that $\boldsymbol{w} \mathrel{\varepsilon\!-} \boldsymbol{z}$. Thus, the requirement that S is internally stable forces $w_1 \neq x_1$ for $\boldsymbol{w} \in S$.

   (b) $w_1 \neq x_1$. By Lemma 1, $\boldsymbol{w}$ and $\boldsymbol{x}$ share a coordinate other than $i = 1$, say $w_2 = x_2$, without loss of generality. Likewise, $\boldsymbol{w}$ and $\boldsymbol{y}$ share a coordinate other than $i = 1, 2$, which implies $w_3 = y_3$. Then, to share a coordinate between $\boldsymbol{w}$ and $\boldsymbol{z}$, either $w_2 = z_2$ or $w_3 = z_3$ must hold, as $w_1 = z_1$ is impossible by $x_1 = z_1$. However, $w_2 = z_2$ implies $x_2 = z_2$, which together with $x_1 = z_1$ imply that $\boldsymbol{x} = \boldsymbol{z}$, a contradiction. The same reasoning is applied for $w_3 = z_3$. Thus, there is no distinct $\boldsymbol{w}$ in S whenever $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$, are collinear.

2. $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, are triangular, as depicted in Figure 2, with $x_1 = y_1$, $y_2 = z_2$, and $z_3 = x_3$.

   Again, assume that there is another distinct $\boldsymbol{w} \in S$. Then:

   (a) $\boldsymbol{w}$ cannot lie on any of the three lines passing through any two of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$. Assume, wlog, that $\boldsymbol{w}$ was collinear with $\boldsymbol{y}$ and $\boldsymbol{z}$; this returns us to the collinear case, above, but with $\boldsymbol{x}$ as the fourth allocation that necessarily violates internal stability.

   (b) it remains only to consider (wlog) $w_2 = x_2$, the dotted line passing through $\boldsymbol{x}$ in Figure 2. By Lemma 1, there must be an agent $i \neq 2$ such that $w_i = y_i$. It cannot be $i = 1$ as that would force, by $x_1 = y_1$, $\boldsymbol{w} = \boldsymbol{x}$. This leaves $w_3 = y_3$, the dotted line through $\boldsymbol{y}$ in Figure 2. Finally, Lemma 1 forces $w_1 = z_1$. The equalities imply

   $$w_1 + w_2 = z_1 + x_2 = 1 - w_3 = 1 - y_3 = y_1 + y_2 = x_1 + z_2.$$

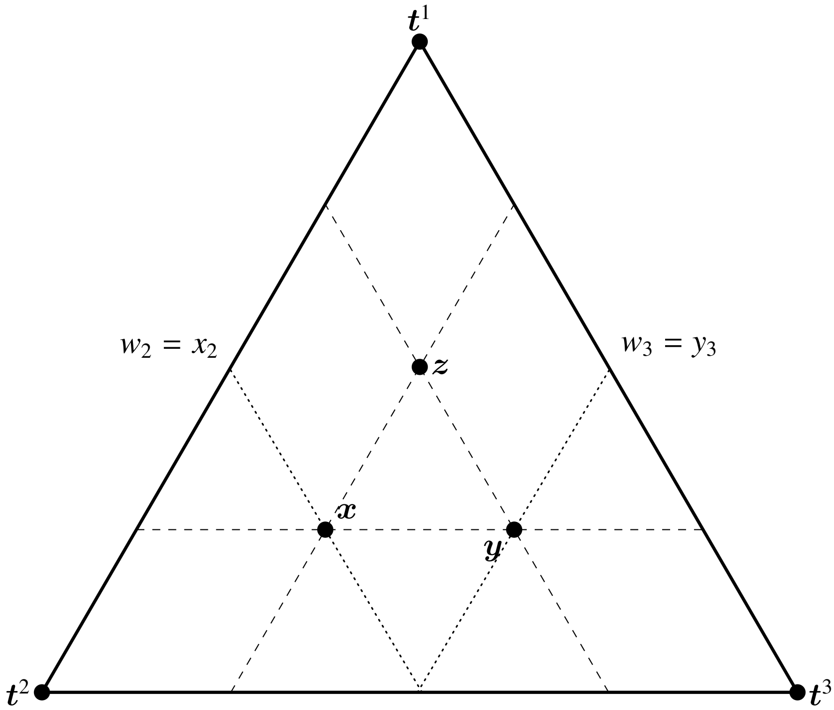

Figure 2: Triangular, internally stable $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ with $x_1 = y_1, y_2 = z_2$ and $z_3 = x_3$

By the second and final terms in the above, we may define

$$k \equiv x_1 - x_2 = z_1 - z_2.$$

As $\boldsymbol{x}$ and $\boldsymbol{z}$ are feasible,

$$x_1 + x_2 + x_3 - k = z_1 + z_2 + z_3 - k \Rightarrow 2x_1 + x_3 = 2z_1 + z_3.$$

Since $z_3 = x_3$, the last equation implies that $x_1 = z_1$, which further implies $\boldsymbol{x} = \boldsymbol{z}$ holds, a contradiction.

Having eliminated the possibility of four internally stable allocations, we may conclude that more than four are also impossible, proving the result. □

Definition 3. *Let $\boldsymbol{s}^{ij}$ be the allocation that* splits *the resource equally between agents i and j, so that $s_i^{ij} = s_j^{ij} = \frac{1}{2}$.*

Thus, unlike the $\boldsymbol{b}^{ij}$ previously defined, the $\boldsymbol{s}^{ij}$ need not balance power.

Then:

Lemma 3. *Suppose that S is stable in a 3-AMPG with an empty core. Then $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.*

Proof. The largest possible S contains, by Lemma 2, three elements, $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$. Consider that possibility first.

By Lemma 1, there are again two possible configurations of three internally stable allocations:

1. *linear*: Without loss of generality, let $\bar{x} \equiv x_1 = y_1 = z_1$, $x_2 > y_2 > z_2$, and $x_3 < y_3 < z_3$, as depicted in Figure 1. We resolve this case in two steps:

   (a) $\boldsymbol{x} = (\bar{x}, 1-\bar{x}, 0)$ *and* $\boldsymbol{z} = (\bar{x}, 0, 1-\bar{x})$. We first show that $\boldsymbol{x} = (\bar{x}, 1-\bar{x}, 0)$. Suppose, on the contrary, that $x_3 > 0$. Then, there exists $\boldsymbol{x}'$ with $x_2 > x_2'$, $x_3' < x_3$, and $x_1' = \bar{x}$. By external stability, this $\boldsymbol{x}'$ must be dominated by one of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. All three possibilities require the same conditions:

   $$\pi(\{3\}, \boldsymbol{x}') > \pi(\{2\}, \boldsymbol{x}') \Leftrightarrow x_3' + v_3 > x_2' + v_2.$$

   However, by internal stability, $\boldsymbol{y} \not\succ \boldsymbol{x}$, which implies $\pi(\{3\}, \boldsymbol{x}) \leq \pi(\{2\}, \boldsymbol{x}) \Leftrightarrow x_3 + v_3 \leq x_2 + v_2$, contradicting $\boldsymbol{x} \succ \boldsymbol{x}'$. Thus, $x_3 = 0$; identical reasoning obtains $z_2 = 0$.

   (b) $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ *cannot satisfy external stability.* Consider $\boldsymbol{x}'' \equiv \frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) + (\varepsilon, 0, -\varepsilon) \in X$ for sufficiently small $\varepsilon > 0$. None of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ dominate $\boldsymbol{x}''$: i. $\boldsymbol{z} \not\succ \boldsymbol{x}''$: dominance would require $\pi(\{3\}, \boldsymbol{x}'') > \pi(\{1,2\}, \boldsymbol{x}'')$, but Theorem 1's empty core property implies that $1 + v_3 < v_1 + v_2$, a contradiction by axiom SR. ii. $\boldsymbol{x} \not\succ \boldsymbol{x}''$ and $\boldsymbol{y} \not\succ \boldsymbol{x}''$: in either case, domination would require $\pi(\{2\}, \boldsymbol{x}'') > \pi(\{1,3\}, \boldsymbol{x}'')$. However, again, by Theorem 1's empty core property with , $1 + v_2 < v_1 + v_3$ holds, contradicting dominance.

2. *triangular*: We resolve this case in three steps:

   (a) *each of* $\boldsymbol{x}, \boldsymbol{y}$ *and* $\boldsymbol{z}$ *must set one component to zero*. Suppose not. Then $\boldsymbol{x} > \boldsymbol{0}$ must hold with $x_1 = y_1$, $y_2 = z_2$, and $z_3 = x_3$, as in Figure 2. Then, consider $\boldsymbol{x}''' \equiv \boldsymbol{x} + (\varepsilon, -2\varepsilon, \varepsilon) \in X$ for sufficiently small $\varepsilon > 0$. None of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ can dominate $\boldsymbol{x}'''$: i. $\boldsymbol{x} \not\succ \boldsymbol{x}'''$: dominance would require $\pi(\{2\}, \boldsymbol{x}''') > \pi(\{1,3\}, \boldsymbol{x}''')$. Again, this contradicts Theorem 1's empty core property, $1 + v_2 < v_1 + v_3$. ii. $\boldsymbol{y} \not\succ \boldsymbol{x}'''$ and $\boldsymbol{z} \not\succ \boldsymbol{x}'''$: in either case, domination would require $W = \{2\}$ and $L = \{1,3\}$ if $x_2 < y_2 = z_2$. Thus, again by the empty core property with Theorem 1, neither $\boldsymbol{y}$ nor $\boldsymbol{z}$ can dominate $\boldsymbol{x}'''$ in this case. Now consider the remaining case, $x_2 > y_2 = z_2$. It implies $x_2''' > y_2 = z_2$ by choosing $\varepsilon > 0$ sufficiently small. Then, $\boldsymbol{y} \succ \boldsymbol{x}'''$ is equivalent to $\pi(\{3\}, \boldsymbol{x}''') > \pi(\{1,2\}, \boldsymbol{x}''')$ with $W = \{3\}$ and $L = \{1,2\}$, while

$\boldsymbol{z} \succ \boldsymbol{x}'''$ is equivalent to $\pi(\{1\}, \boldsymbol{x}''') > \pi(\{2,3\}, \boldsymbol{x}''')$ with $W = \{1\}$ and $L = \{2,3\}$. Either case is impossible by the empty core property with Theorem 1. In summary, there exists $\boldsymbol{x}''' \in X$, undominated by any of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, which contradicts the external stability of S.

(b) $S \subseteq \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$. By the previous step and Lemma 1, we only need to consider the following two possibilities:

i. $\boldsymbol{x} = (a, 0, 1-a)$, $\boldsymbol{y} = (a, 1-a, 0)$, and $\boldsymbol{z} = (0, a, 1-a)$.

ii. $\boldsymbol{x} = \boldsymbol{t}^1$, $\boldsymbol{y} = \boldsymbol{t}^2$, and $\boldsymbol{z} = \boldsymbol{t}^3$.

The second case violates external stability: for any feasible $\boldsymbol{w} > \boldsymbol{0}$ to be dominated by one of $\boldsymbol{t}^1, \boldsymbol{t}^2$, and $\boldsymbol{t}^3$, it must be that $\#W = 1$, $\#L = 2$, and $\pi(W, \boldsymbol{w}) > \pi(L, \boldsymbol{w})$. However, by axiom SR, the last inequality is impossible by the empty core property.

For the first case, the resource constraint requires $a = \frac{1}{2}$, so that $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\} = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

(c) $S \supseteq \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$. Suppose, without loss of generality, $\boldsymbol{s}^{23} \notin S$. Then, consider $\boldsymbol{w} \equiv \frac{1}{2}\left(\boldsymbol{s}^{12} + \boldsymbol{s}^{13}\right) + (\varepsilon, 0, -\varepsilon) \in X$ for sufficiently small $\varepsilon > 0$. By the reasoning developed in the external stability step of the linear case above, neither $\boldsymbol{s}^{12}$ nor $\boldsymbol{s}^{13}$ can dominate $\boldsymbol{w}$ due to Theorem 1's empty core property.

□

Intuitively, Lemma 1 allowed only linear or triangular three-element stable sets. Linear sets do not dominate some non-collinear allocations, violating external stability. In triangular sets, each allocation sets at least one term to zero: if not, they leave undominated more extremal allocations; the equal split configuration then also ensures that the allocations between stable elements are dominated.

The main result of this section generalises Jordan and Obadia (2015, Theorem 3.7) beyond the symmetric case:

Theorem 2. *For a* 3*-AMPG with an empty core, $v_a - v_c \leq \frac{1}{2}$ is necessary and sufficient for internal stability of $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$. Empty core condition* (6) *suffices for the external stability of S, leaving it the unique stable set when it is internally stable.*

Proof. By Lemma 3, the only candidate stable set is $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

Internal stability implies $v_i \leq \frac{1}{2} + v_j$ for all i and j. By inequality (3)'s ordering, this is equivalent to $v_a \leq \frac{1}{2} + v_c$, the lemma's stated inequality. Now establish the other direction, that this inequality implies internal stability. By the resource monotonicity axioms, it must be that $\boldsymbol{s}^{ab} \not\succ \boldsymbol{s}^{bc}$: the agent with the greatest intrinsic strength (but no resources) cannot defeat that with the least intrinsic strength (but with half the resources); this is equivalent to the lemma's inequality.

By external stability, each $\boldsymbol{x} \in X \backslash S$ that is undominated by an $\boldsymbol{s} \in S$ implies that there are distinct j and k such that $x_j, x_k < \frac{1}{2}$. By inequality (3), empty core inequality (6) implies that $1 + v_i < v_j + v_k$ for agents j and k. As $x_i \leq 1$ and $x_j, x_k \geq 0$, it follows that $x_i + v_i < x_j + v_j + x_k + v_k$, so that $\boldsymbol{s}^{jk} \succ \boldsymbol{x}$, establishing external stability. □

Perhaps surprisingly, the allocations in the stable set do not vary with v_i until the set ceases to exist. Intuitively, the empty core condition limits the relative strength of the intrinsically strongest agent. This prevents any single agent overpowering the other two for any allocation. As a result, the only power contests that must be considered are those between singleton coalitions. The theorem's bound on relative power ensures that the intrinsically least powerful agent with half of the resource can defend itself against the most powerful with none of the resource. Thus, power need not be equally balanced at the $\boldsymbol{s}^{ij}$ allocations: while v_a increases relative to v_c, there is no distortion in S until the asymmetry becomes too large, and internal stability fails completely.

4 The non-empty core

This section derives the stable sets (if any) corresponding to each of the possible non-empty cores identified in Corollary 1.

As the core must be included in any stable set, the proofs follow a similar pattern: the core seeds a candidate stable set; the allocations dominated by the core are excluded from consideration; over the remaining allocations (often just linear loci) it is as if the core is empty, allowing the use of techniques from the previous section. Of particular importance is the locus of allocations along which the most intrinsically powerful agent, a, is just as powerful as the other two, b, c; we prove that the existence of a stable set on X requires the existence of a stable set along that locus. When non-existence arises, it is for the same reason identified in the symmetric case in Rowat and Kerber (2014): allocations on these loci are only dominated by more extreme ones; thus, external stability requires including the most extreme allocations in a stable set; when these extreme allocations are themselves dominated by a core allocation, existence fails.

Define the balance of power locus for agent i to be:

$$B(i) \equiv \{\boldsymbol{x} \in X \,|\, \pi(\{i\}, \boldsymbol{x}) = \pi(\{j,k\}, \boldsymbol{x})\} \tag{16}$$

The following lemma helps establish the result that existence of a stable set on X requires the existence of a stable set on non-empty balance of power loci:

Lemma 4. *Consider a* 3*-AMPG with non-empty core. Then, for each* $i \in I$ *such that* $B(i) \neq \emptyset$, $\boldsymbol{x} \in B(i) \Rightarrow \nexists \boldsymbol{y} \in X \backslash B(i)$ *such that* $y_i < x_i$ *and* $\boldsymbol{y} \succ \boldsymbol{x}$.

Proof. As $y_i < x_i$, agent i would oppose $\boldsymbol{y} \succ \boldsymbol{x}$. By definition, $\boldsymbol{x} \in B(i)$ implies that agent i on its own is as strong as agents j and k together. By axiom WC, this is the most powerful coalition that can oppose i, concluding the proof. □

This helps us decompose the search for stable sets: for all non-empty $B(i)$, if a stable set exists, its intersection with $B(i)$ must be a stable set on $B(i)$:

Corollary 2. *In a* 3*-AMPG with a non-empty core, if a stable set* S *exists in* X*, then* $S \cap B(i) \neq \emptyset$ *and* $D(S \cap B(i)) \supseteq B(i) \backslash S$ *hold for each* $i \in I$ *with* $B(i) \neq \emptyset$.

Proof. For external stability to hold, each $\boldsymbol{x} \in B(i)$ must either belong to a stable set S or be dominated by an allocation in S. We will show that if the latter case applies, then that allocation in S also belongs to $B(i)$.

If $x_i = 1$ for $\boldsymbol{x} \in B(i)$, this implies $\boldsymbol{x} = \boldsymbol{t}^i$. As $\boldsymbol{x} = \boldsymbol{t}^i \in B(i)$, $v_i + 1 = v_j + v_k$ holds, which implies by Theorem 1, $\boldsymbol{t}^i \in K$. Thus, $\boldsymbol{x} \in S$ holds.

If $x_i \in [0, 1)$ for $\boldsymbol{x} \in B(i)$, then consider the following three subcases:

1. Let $\boldsymbol{z} \in X$ be such that $z_i > x_i$. By $\boldsymbol{x} \in B(i)$ and axiom SR, $\pi(\{i\}, \boldsymbol{z}) > \pi(\{j, k\}, \boldsymbol{z})$, which implies that $\boldsymbol{t}^i \succ \boldsymbol{z}$. Moreover, the last strict inequality implies that $1 + v_i > v_j + v_k$, and thus $\boldsymbol{t}^i \in K \subseteq S$ by Theorem 1. Therefore, by the internal stability condition, $\boldsymbol{z} \notin S$ holds. Therefore, $\boldsymbol{z}$ cannot be used to show that $\boldsymbol{x} \notin S$.

2. Let $\boldsymbol{y} \in X$ be such that $y_i < x_i$. By Lemma 4, $\boldsymbol{y} \not\succ \boldsymbol{x}$. Again, $\boldsymbol{y}$ cannot be used to show that $\boldsymbol{x} \notin S$.

3. By the previous two subcases, if $\boldsymbol{x} \notin S$, then there must exist $\boldsymbol{x}' \in S$ such that both $x'_i = x_i$ and $\boldsymbol{x}' \succ \boldsymbol{x}$ hold. By definition, $\boldsymbol{x}' \in B(i)$.

Thus, for each $i \in I$ with $B(i) \neq \emptyset$, $S \cap B(i) \neq \emptyset$ and $D(S \cap B(i)) \supseteq B(i) \backslash S$ hold. □

Some elements in the $B(i)$ play an important role in determining whether stable sets exist. First:

$$\beta_i^i \equiv \frac{v_j + v_k - v_i + 1}{2}; \tag{17}$$

$$\boldsymbol{e}_b^a \equiv (\beta_a^a, 1 - \beta_a^a, 0)\,; \boldsymbol{e}_a^b \equiv \left(1 - \beta_b^b, \beta_b^b, 0\right); \text{ and } \boldsymbol{e}_a^c \equiv (1 - \beta_c^c, 0, \beta_c^c)\,.$$

Define $\bar{\boldsymbol{\beta}}^i$ as the allocation on $B(i)$ that balances the power of j and k, so that $\bar{\beta}_j^i + v_j = \bar{\beta}_k^i + v_k$:

$$\bar{\boldsymbol{\beta}}^i \equiv \left(\beta_i^i, \bar{\beta}_j^i, \bar{\beta}_k^i\right),$$

where

$$\bar{\beta}^i_j \equiv \frac{1 + v_i + v_k - 3v_j}{4}; \text{ and } \bar{\beta}^i_k \equiv \frac{1 + v_i + v_j - 3v_k}{4}.$$

Assuming $v_j \geq v_k$ without loss of generality, it follows that

$$\bar{\boldsymbol{\beta}}^i \in X \Leftrightarrow \beta^i_i \in [0, 1]\,; \text{ and } \left(1 - \beta^i_i\right) \geq \left|v_k - v_j\right|.$$

By definition,

$$B(i) \neq \emptyset \text{ if and only if } \beta^i_i \in [0, 1]\,.$$

Corollary 3. *In a* 3*-AMPG with a non-empty core, let a stable set S exist in X. Then,*

$$B(a) \neq \emptyset \Rightarrow \boldsymbol{e}^a_b \in S\,; B(b) \neq \emptyset \Rightarrow \boldsymbol{e}^b_a \in S\,; \textit{ and } B(c) \neq \emptyset \Rightarrow \boldsymbol{e}^c_a \in S.$$

Moreover,

$$\begin{aligned} &B(a) \neq \emptyset, \bar{\boldsymbol{\beta}}^a \in X \Rightarrow \boldsymbol{e}^a_c, \bar{\boldsymbol{\beta}}^a \in S\,;\\ &B(b) \neq \emptyset, \bar{\boldsymbol{\beta}}^b \in X \Rightarrow \boldsymbol{e}^b_c, \bar{\boldsymbol{\beta}}^b \in S\,; \textit{ and}\\ &B(c) \neq \emptyset, \bar{\boldsymbol{\beta}}^c \in X \Rightarrow \boldsymbol{e}^c_b, \bar{\boldsymbol{\beta}}^b \in S. \end{aligned}$$

Before proving the Corollary, we provide some background for the non-existence mechanism (q.v. also Rowat and Kerber (2014)):

Definition 4. *The dominance operator,* $\succ$ *, is a* strict total order *on a set of allocations, A, iff, for any* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in A$ *the following hold:*

1. *trichotomy: exactly one of* $\boldsymbol{x} = \boldsymbol{y}$*,* $\boldsymbol{x} \succ \boldsymbol{y}$ *and* $\boldsymbol{y} \succ \boldsymbol{x}$ *holds; and*
2. *transitivity:* $\boldsymbol{x} \succ \boldsymbol{y} \succ \boldsymbol{z}$ *implies that* $\boldsymbol{x} \succ \boldsymbol{z}$*.*

This is also called a *complete ordering* (von Neumann and Morgenstern, 1953, §65.3.1). For concision's sake, we adopt their terminology.

Before applying these concepts, first define:

Definition 5. *Given* $B(i) \neq \emptyset$ *with* $i, j, k \in I$ *such that* $v_j \geq v_k$*, let*

$$\begin{aligned} \left[\boldsymbol{e}^i_j, \bar{\boldsymbol{\beta}}^i\right] &\equiv \left\{x = \left(\beta^i_i, x_j, x_k\right) \in B(i) \mid \bar{\beta}^i_j \leq x_j \leq 1 - \beta^i_i, \bar{\beta}^i_k \geq x_k \geq 0, x_j + x_k = 1 - \beta^i_i\right\};\\ \left[\boldsymbol{e}^i_j, \boldsymbol{e}^i_k\right] &\equiv \left\{x = \left(\beta^i_i, x_j, x_k\right) \in B(i) \mid 0 \leq x_j \leq 1 - \beta^i_i, 1 - \beta^i_i \geq x_k \geq 0, x_j + x_k = 1 - \beta^i_i\right\}. \end{aligned}$$

Thus, whether or not $\bar{\boldsymbol{\beta}}^i \in X$, $B(i) = \left[\boldsymbol{e}^i_j, \boldsymbol{e}^i_k\right]$ holds whenever $B(i) \neq \emptyset$.

Lemma 5. *In a* 3*-AMPG, the dominance operator* $\succ$ *completely orders the locus of allocations over the subsets of X specified below:*

1. *for* $B(a) \neq \varnothing$: $\left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right) \bigcup \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$ *if* $\bar{\boldsymbol{\beta}}^a \in X$, *and* $\left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$ *otherwise;*

2. *for* $B(b) \neq \varnothing$: $\left[\boldsymbol{e}^b_a, \bar{\boldsymbol{\beta}}^b\right) \bigcup \left(\bar{\boldsymbol{\beta}}^b, \boldsymbol{e}^b_c\right]$ *if* $\bar{\boldsymbol{\beta}}^b \in X$, *and* $\left[\boldsymbol{e}^b_a, \boldsymbol{e}^b_c\right]$ *otherwise;*

3. *for* $B(c) \neq \varnothing$: $\left[\boldsymbol{e}^c_a, \bar{\boldsymbol{\beta}}^c\right) \bigcup \left(\bar{\boldsymbol{\beta}}^c, \boldsymbol{e}^c_b\right]$ *if* $\bar{\boldsymbol{\beta}}^c \in X$, *and* $\left[\boldsymbol{e}^c_a, \boldsymbol{e}^c_b\right]$ *otherwise.*

Proof. Let $B(a) \neq \varnothing$ and suppose that $\bar{\boldsymbol{\beta}}^a \in X$.

Consider $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$. Then, as $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in B(a)$, $x_a = y_a = z_a = \beta^a_a$. Thus, if $x_b = y_b$, then by the unit endowment, $\boldsymbol{x} = \boldsymbol{y}$ holds. If $x_b > y_b$, then since $\bar{\beta}^a_b < x_b, y_b \leq 1 - \beta^a_a$ over $\left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$, it follows that $y_b + v_b > y_c + v_c$, so that $\boldsymbol{x} \mathrel{\text{⋲}} \boldsymbol{y}$. This satisfies trichotomy over $\left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$. As dominance on this locus is equivalent to $x_b > y_b$, transitivity over $\left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$ follows from that of strict inequality.

Now consider $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$. As before, $x_a = y_a = z_a = \beta^a_a$. Again, if $x_b = y_b$, then $\boldsymbol{x} = \boldsymbol{y}$. If $x_b > y_b$, it implies $x_c < y_c$. Then, since $\bar{\beta}^a_c < x_c, y_c \leq 1 - \beta^a_a$ over $\left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$, it follows that $x_c + v_c > x_b + v_b$, so that $\boldsymbol{y} \mathrel{\text{⋲}} \boldsymbol{x}$. This satisfies trichotomy over $\left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$. As dominance on this locus is equivalent to $y_c > x_c$, transitivity over $\left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$ follows from that of strict inequality.

Finally, suppose $\bar{\boldsymbol{\beta}}^a \notin X$. Consider $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$. Note that $\bar{\boldsymbol{\beta}}^a \notin X$ while $B(a) \neq \varnothing$ implies that $(1 - \beta^a_a) < |v_c - v_b|$, which in turn implies $\bar{\beta}^a_b < 0$. Thus, it implies $v_b > (1 - \beta^a_a) + v_c$. Hence, for any $\boldsymbol{w} \in \left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$, $w_b + v_b > w_c + v_c$ holds. Therefore, if $x_b > y_b$, then $y_b + v_b > y_c + v_c$, and thus $\boldsymbol{x} \mathrel{\text{⋲}} \boldsymbol{y}$ holds. This shows that $\mathrel{\text{⋲}}$ satisfies trichotomy and transitivity over $\left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$ in the same way as the above two cases.

For $B(b) \neq \varnothing$ and $B(c) \neq \varnothing$, the same reasoning applies. □

Proof of Corollary 3. Consider $B(a) \neq \varnothing$ and $\bar{\boldsymbol{\beta}}^a \notin X$. As a stable set S exists, Corollary 2 implies that $B(a) \cap S \neq \varnothing$. Suppose $\boldsymbol{e}^a_b \notin S$. Then there must exist $\boldsymbol{x} \in B(a) \cap S$ such that $\boldsymbol{x} \mathrel{\text{⋲}} \boldsymbol{e}^a_b$. In a pillage from $\boldsymbol{e}^a_b$ to $\boldsymbol{x}$, $W = \{c\}$ and $L = \{b\}$ hold. However, $B(a) = \left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$ and Lemma 5 together imply both that the dominance operator $\mathrel{\text{⋲}}$ completely orders over $B(a)$, and that $\boldsymbol{y} \mathrel{\text{⋲}} \boldsymbol{z} \Leftrightarrow y_b > z_b$ holds for any $\boldsymbol{y}, \boldsymbol{z} \in \left[\boldsymbol{e}^a_b, \boldsymbol{e}^a_c\right]$. Therefore, $\boldsymbol{x} \not\mathrel{\text{⋲}} \boldsymbol{e}^a_b$, a contradiction. Therefore, $\boldsymbol{e}^a_b \in S$.

Second, consider $B(a) \neq \varnothing$ and $\bar{\boldsymbol{\beta}}^a \in X$. Then, via a symmetric argument to the previous case, it can be shown that $\boldsymbol{e}^a_b \in S$ by the proof of Corollary 2, $\boldsymbol{e}^a_b \in \left[\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$, and $(1 - \beta^a_a) + v_b > v_c$. Likewise, $\boldsymbol{e}^a_c \in S$ by the proof of Corollary 2, $\boldsymbol{e}^a_c \in \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right]$, and $(1 - \beta^a_a) + v_c > v_b$.

Finally, suppose $\bar{\boldsymbol{\beta}}^a \notin S$. Then, there must exist $\boldsymbol{x} \in B(a) \cap S$ such that $\boldsymbol{x} \mathrel{\text{⋲}} \bar{\boldsymbol{\beta}}^a$. However, as $v_b + \bar{\beta}^a_b = v_c + \bar{\beta}^a_c$, $\boldsymbol{x} \not\mathrel{\text{⋲}} \bar{\boldsymbol{\beta}}^a$ holds, which is a contradiction. Therefore, $\bar{\boldsymbol{\beta}}^a \in S$.

For the cases of $B(b) \neq \varnothing$ and $B(c) \neq \varnothing$, the same arguments can be applied. $\square$

We may now present non-existence conditions directly on the parameters v_a, v_b and v_c. In the following, the parameter conditions ensure that the maximal elements in the $B(i)$ are dominated by a core element:

Corollary 4. *In a 3-AMPG with a non-empty core, let $v_j \geq v_k$, without loss of generality. Let $-v_i + v_j + v_k \in [-1, 1)$ and one of the following two conditions be satisfied:*

1. $v_k > 0$*; or*
2. $v_j > 0$ *and* $1 + v_i - 3v_j + v_k > 0$.

Then, no stable set exists.

Proof. By $-v_i + v_j + v_k \in [-1, 1)$, $0 \leq \beta_i^i < 1$ holds, hence $B(i) \neq \varnothing$. Assume, on the contrary, that a stable set S exists. Note also that $-v_i + v_j + v_k < 1$ implies that $\boldsymbol{t}^i \in K$ by Theorem 1. Then, as $S \neq \varnothing$ by assumption, $\boldsymbol{t}^i \in S$.

Consider the first inequality in the corollary's statement. By Corollary 3, $S \neq \varnothing$ and $B(i) \neq \varnothing$ imply that $\boldsymbol{e}_j^i \in S$. However, $v_i + \beta_i^i > v_j + \left(1 - \beta_i^i\right)$ follows from $v_k > 0$, which implies that $\boldsymbol{t}^i \succ \boldsymbol{e}_j^i$. This is a contradiction. Therefore, when the first inequality holds, no stable set exists.

Now consider the second. Then, as $1 + v_i - 3v_j + v_k > 0$ implies $\left(1 - \beta_i^i\right) > \left|v_k - v_j\right|$, and so $\bar{\boldsymbol{\beta}}^i \in X$ which is also an interior allocation in X. Then, by Corollary 3, $S \neq \varnothing$ and $B(i) \neq \varnothing$ imply that $\boldsymbol{e}_k^i \in S$. However, $v_i + \beta_i^i > v_k + \left(1 - \beta_i^i\right)$ follows from $v_j > 0$. Then, by the same reasoning as in the first case, we derive a contradiction. Thus, in this case as well, no stable set exists. $\square$

The remainder of this section characterises stable sets for each of the non-empty core cases identified in Corollary 1 when $n = 3$. While the corollary worked directly with parameter values for $\boldsymbol{v}$, the theorems work with geometric objects in the simplex, X, to aid intuitions. Thus, following statement of the theorems, Table 1 provides a mapping between corollary's parameter ranges the theorems' geometric objects. Proofs are in the appendix.

Theorem 3. *Consider a 3-AMPG with $K = \{\boldsymbol{t}^a\}$. Then:*

1. *if $B(a) = \varnothing$, then there exists the unique stable set $S = \{\boldsymbol{t}^a\}$.*
2. *if $B(a) = \{\boldsymbol{t}^a\}$, then no stable set exists.*
3. *if $\boldsymbol{t}^a \notin B(a) \neq \varnothing$, then $\boldsymbol{e}_b^a \neq \boldsymbol{b}^{ab}$ holds and no stable set exists.*

Theorem 4. *Consider a* 3*-AMPG with* $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$*. Then,* $B(a) \neq \emptyset$*, and:*

1. *there exists the unique stable set* $S = K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$ *if and only if* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab}$ *and* $\bar{\boldsymbol{\beta}}^a \notin X$ *hold.*
2. *otherwise, no stable set exists.*

When $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab}$*,* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$ *holds.*

Theorem 5. *Consider a* 3*-AMPG with* $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$*. Then,* $B(a) \neq \emptyset$*,* $B(b) \neq \emptyset$*, and* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{e}^b_a$ *hold, and:*

1. *there exists the unique stable set* $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$ *if and only if* $\bar{\boldsymbol{\beta}}^a \notin X$ *holds.*
2. *otherwise, no stable set exists.*

Theorem 6. *Consider a* 3*-AMPG with* $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$*. Then,* $B(a) \neq \emptyset$*,* $B(b) \neq \emptyset$*, and* $B(c) \neq \emptyset$ *hold. Moreover, the following statements hold:*

1. *Let* $v_c = 0$*. Then,* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$ *and* $\boldsymbol{e}^a_c = \boldsymbol{b}^{ac} = \boldsymbol{t}^c$ *hold, and there exists the unique stable set*

$$S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{bc}\right\} = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{s}^{bc}\right\} = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \bar{\boldsymbol{\beta}}^a\right\}.$$

2. *Let* $v_c > 0$*. Then,* $B(a) = \{\boldsymbol{t}^a\}$*,* $B(b) = \left\{\boldsymbol{t}^b\right\}$*,* $B(c) = \{\boldsymbol{t}^c\}$*, and there exists the unique stable set*

$$S = K \cup \left\{\boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\} = K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$$

   if and only if $v_a = v_b = v_c = 1$ *holds. Otherwise, no stable set exists.*

Theorem 7. *Consider a* 3*-AMPG with* $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$*. Then,* $B(a) \neq \emptyset$*,* $B(b) \neq \emptyset$*,* $B(c) \neq \emptyset$*,* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{e}^b_a$ *hold, and:*

1. *there exists the unique stable set* $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$ *if and only if* $v_a = v_b = \frac{1}{2}$ *holds.*
2. *otherwise, no stable set exists.*

Theorem 8. *Consider a* 3*-AMPG with* $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}\right\}$*. Then,* $B(a) \neq \emptyset$*,* $B(b) \neq \emptyset$*,* $B(c) \neq \emptyset$*,* $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{e}^b_a$*,* $\boldsymbol{e}^a_c = \boldsymbol{b}^{ac} = \boldsymbol{e}^c_a$ *, and* $v_a < 1$ *hold. Moreover:*

1. *there exists the unique stable set* $S = K \cup \bar{\boldsymbol{\beta}}^a$ *if and only if* $v_a \geq \frac{1}{3}$ *holds.*

2. *otherwise, no stable set exists.*

Finally, the case in which $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\}$, is the symmetric *wealth-is-power* case (Jordan, 2006). In the present $n = 3$ case, the stable set — illustrated in Figure 3 — reduces to:

Theorem 9. *Jordan (2006, Theorem 3.3) In the wealth-is-power pillage game with $n = 3$ and $v_i = 0$ for all $i = 1, 2, 3$, the unique stable set is*

$$S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}, \bar{\boldsymbol{\beta}}^a, \bar{\boldsymbol{\beta}}^b, \bar{\boldsymbol{\beta}}^c\right\}.$$

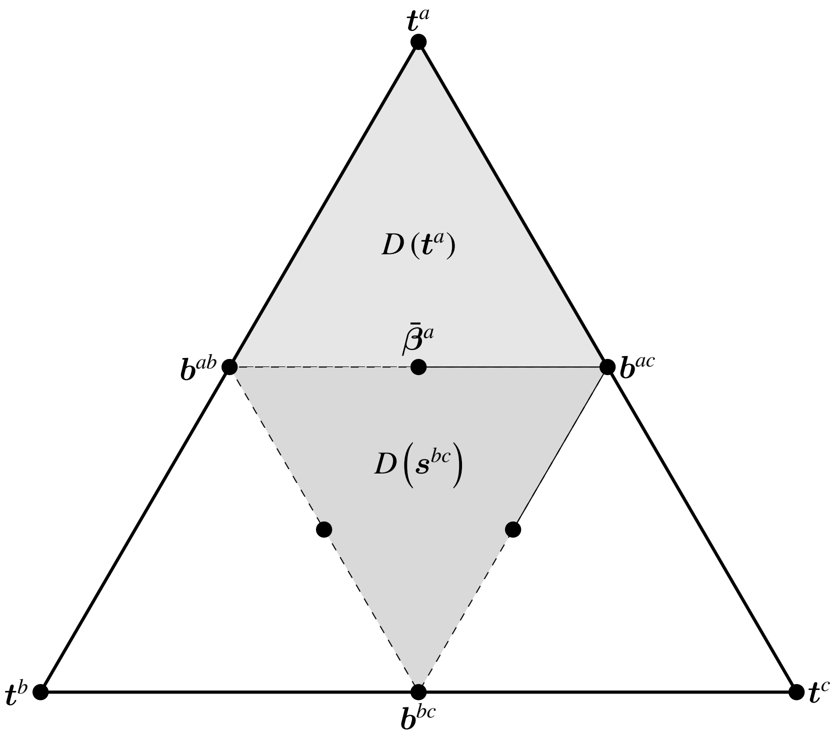

Figure 3: Wealth-is-power: $\pi(C, \boldsymbol{x}) = \sum_{i \in C} x_i$

Table 1 maps between Corollary 1's inequalities and the preceding theorems.

5 Discussion

This paper removes the symmetry assumption usually assumed in analysis of pillage games. It is motivated by a hope of identifying a tractable class of pillage games to allow empirical tests of this theory of unstructured power contests.

inequality	secondary conditions	key elements	stable set, S	theorem
(7)	$-v_a + v_b + v_c < -1$	$B(a) = \varnothing$	K	3
	$-v_a + v_b + v_c \in [-1, 1)$	$\boldsymbol{t}^a \notin B(a) \neq \varnothing, \boldsymbol{e}_b^a \neq \boldsymbol{b}^{ab}$	$\nexists$	
	$-v_a + v_b + v_c = 1$	$B(a) = \{\boldsymbol{t}^a\}$	$\nexists$	
(8)	n.a.	$B(a) \neq \varnothing, \boldsymbol{e}_b^a \neq \boldsymbol{b}^{ab}$	$\nexists$	4
(9)		$B(a) \neq \varnothing, \boldsymbol{e}_b^a = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$		
	$v_a > 2$	$\bar{\boldsymbol{\beta}}^a \notin X$	K	
	$v_a \in (1, 2]$	$\bar{\boldsymbol{\beta}}^a \in X$	$\nexists$	
(10)		$B(a) \neq \varnothing, B(b) \neq \varnothing, \boldsymbol{b}^{ab} = \boldsymbol{e}_b^a = \boldsymbol{e}_a^b$		5
	$v_b \geq \frac{1}{3}(1 + v_a)$	$\bar{\boldsymbol{\beta}}^a \notin X$	K	
	$v_b < \frac{1}{3}(1 + v_a)$	$\bar{\boldsymbol{\beta}}^a \in X$	$\nexists$	
(11)	$v_a = v_b = v_c = 1$	$\boldsymbol{e}_b^a = \boldsymbol{t}^a, \boldsymbol{e}_a^b = \boldsymbol{t}^b, \boldsymbol{e}_a^c = \boldsymbol{t}^c, B(i) = \left\{\boldsymbol{t}^i\right\} \forall i \in I$	$K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$	6
	otherwise	$B(i) \neq \varnothing \forall i \in I$	$\nexists$	
(12)	n.a.	$B(a) = \bar{X}_{bc}, \boldsymbol{e}_b^a = \boldsymbol{b}^{ab} = \boldsymbol{t}^b, \bar{\boldsymbol{\beta}}^a = \left(0, \frac{1}{2}, \frac{1}{2}\right) = \boldsymbol{b}^{bc} = \boldsymbol{s}^{bc}$	$K \cup \left\{\boldsymbol{s}^{bc}\right\}$	
(13)		$\boldsymbol{e}_b^a = \boldsymbol{b}^{ab} = \boldsymbol{e}_a^b$		7
	$v_a = v_b = \frac{1}{2}$	$B(i) = \left\{\boldsymbol{x} \in X \mid x_i = \frac{1}{2}\right\} \forall i \in \{a, b\}, B(c) = \{\boldsymbol{t}^c\}$	K	
	otherwise	$B(a) = \left\{\boldsymbol{x} \in X \mid x_a = \frac{v_b - v_a + 1}{2}\right\}, B(b) = \left\{\boldsymbol{x} \in X \mid x_b = \frac{v_a - v_b + 1}{2}\right\}, B(c) = \left\{\boldsymbol{x} \in X \mid x_c = \frac{v_a + v_b + 1}{2}\right\}$	$\nexists$	
(14)	$v_a \geq \frac{1}{3}$	$\boldsymbol{e}_b^a = \boldsymbol{b}^{ab} = \boldsymbol{e}_a^b, \boldsymbol{e}_c^a = \boldsymbol{b}^{ac} = \boldsymbol{e}_a^c$		8
		$v_a = \frac{1}{3} \Rightarrow \bar{\boldsymbol{\beta}}^b = \boldsymbol{e}_c^b, \bar{\boldsymbol{\beta}}^c = \boldsymbol{e}_b^c$	$K \cup \left\{\bar{\boldsymbol{\beta}}^a\right\}$	
		$v_a \in \left(\frac{1}{3}, 1\right) \Rightarrow \bar{\boldsymbol{\beta}}^a = \left(\frac{1 - v_a}{2}, \frac{1 + v_a}{4}, \frac{1 + v_a}{4}\right)$	$K \cup \left\{\bar{\boldsymbol{\beta}}^a\right\}$	
	otherwise	$B(i) \neq \varnothing \forall i \in I$	$\nexists$	
(15)	n.a.	$\boldsymbol{e}_b^a = \boldsymbol{b}^{ab} = \boldsymbol{e}_a^b, \boldsymbol{e}_c^a = \boldsymbol{b}^{ac} = \boldsymbol{e}_a^c, \boldsymbol{e}_c^b = \boldsymbol{b}^{bc} = \boldsymbol{e}_b^c, \bar{\boldsymbol{\beta}}^a = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \bar{\boldsymbol{\beta}}^b = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right), \bar{\boldsymbol{\beta}}^c = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$	$K \cup \left\{\bar{\boldsymbol{\beta}}^a, \bar{\boldsymbol{\beta}}^b, \bar{\boldsymbol{\beta}}^c\right\}$	9

Table 1: Mapping between Corollary 1 parameters and Theorems 3 – 9 features

It finds, first, that stable sets are unique — when they exist — in $n = 3$ AMPG. MacKenzie, Kerber, and Rowat (2015) demonstrated that violating the symmetry axiom could lead to multiple stable sets in majority pillage games when the number of agents was at least four. This paper therefore establishes that as a lower bound for multiplicity in AMPG.

Second, it suggests a link between linear power functions and winner(s)-take-all contests: the only case in which the stable set contains interior allocations, $\boldsymbol{x} > \mathbf{0}$, are the special cases of $\boldsymbol{v} = \mathbf{0}$ and $v_a \in \left[\frac{1}{3}, 1\right), v_b = v_c = 0$. This reflects the tension between existence of a stable set and strictly interior allocations along balance of power loci: when there is an interior allocation at which the two contestants' power is balanced, its extremal elements must also belong to a stable set; if, though, the relevant $v_i > 0$, these elements may be dominated by a tyrannical allocation, preventing existence.

Third, it finds that non-existence is pervasive. Thus, the theory fails to deliver testable predictions for non-negligible sets of values of v_1, v_2 and v_3.

Tantalisingly, the theory yields predictions for almost the full range of the special cases noted above, for which $v_b = v_c = 0$ and $v_a > 0$ — so that agent a has some source of intrinsic strength that b and c do not. This power could arise from greater internal cohesion, a structural advantage (e.g. perhaps being located more centrally). Figure 4 illustrates, with v_a increasing from 0 in the leftmost diagram to $v_a > 1$ in the rightmost. Filled dots represent allocations in a stable set.

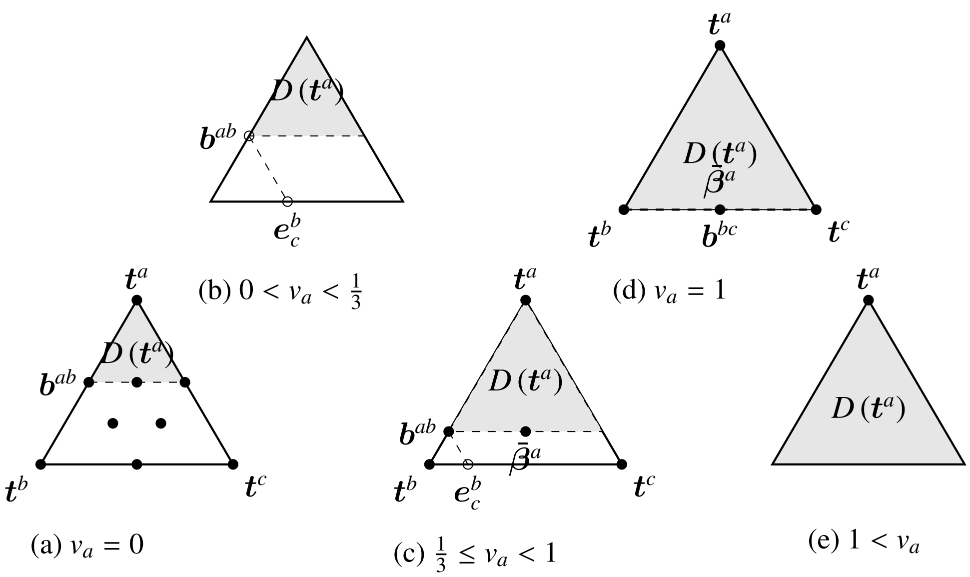

Figure 4: Stable sets when $v_a \geq v_b = v_c = 0$

When $v_a = 0$, the fully symmetric case depicted in diagram 4a, a stable set exists with interior allocations, as per Theorem 3.3 of Jordan (2006). As v_a increases, $D(\boldsymbol{t}^a)$ reaches further into the simplex, as depicted in diagram 4b, so that $\boldsymbol{b}^{ab}$ no longer dominates the extremal $\boldsymbol{e}_c^b$; no stable set exists, as per the second case in Theorem 8. When v_a grows beyond $\frac{1}{3}$ (q.v. diagram 4c), $\boldsymbol{b}^{ab}$ comes to dominate all allocations to the extremal $\boldsymbol{e}_c^b$; this allows the stable set in the first case of Theorem 8. When $v_a = 1$, $D(\boldsymbol{t}^a)$ pushes further down, dominating the whole simplex except its bottom margin (as depicted in diagram 4d); now the stable set in the second case of Theorem 6 exists. Finally, when $v_a > 1$, depicted in diagram 4e, $\boldsymbol{t}^a$ dominates the whole simplex, leaving it the singleton stable set, as per the first case of Theorem 3.

Thus, the cardinality of the unique stable set decreases in v_a, with a gap at $v_a \in \left(0, \frac{1}{3}\right)$, for which no stable set exists.

We conclude by mentioning three possible ways forward for an empirically-testable theory of pillage games.

First, production could be considered, so that assigning an agent nothing inefficiently excludes its production function from society. In the terminology of Olson (1993), production provides an incentive for bandits to be stationary rather than roving. Production in pillage games was first considered by Jordan (2009), which gave each agent in a pillage game a production function for converting wealth into consumption goods. While production could be expected to favour interior allocations, it is less obvious that they would rescue existence.

Second, the tension between existence and interior allocations arises in part because the balance of power loci (e.g. Figure 3's dashed lines) are linear: interior allocations do not dominate their neighbours as the loci restrict power contests to those pitting one agent against another. However, a power function like

$$\pi(C, \boldsymbol{x}) = \sum_{i \in C} \left(\sqrt{x_i} + v_i \right);$$

generates curved balance of power loci which allow the third agent (who would be indifferent along a linear balance of power locus) to benefit from moves towards the centre of the locus. Its involvement allows central allocations to dominate the extremes of the locus, rescuing existence and yielding strictly interior solutions. Figure 5 illustrates the symmetric special case in which $v_i = 0$ for all agents. Its unique stable set contains strictly interior allocations:

$$S = \left\{ \boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}, \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right) \right\}.$$

For $n > 3$, we conjecture that, in the empty core case, the results in the symmetric case studied by Jordan and Obadia (2015) can be extended to the asymmetric case with an additional condition ensuring that any larger coalition dominates any smaller coalition: for even n, no symmetric stable set should exist; for odd

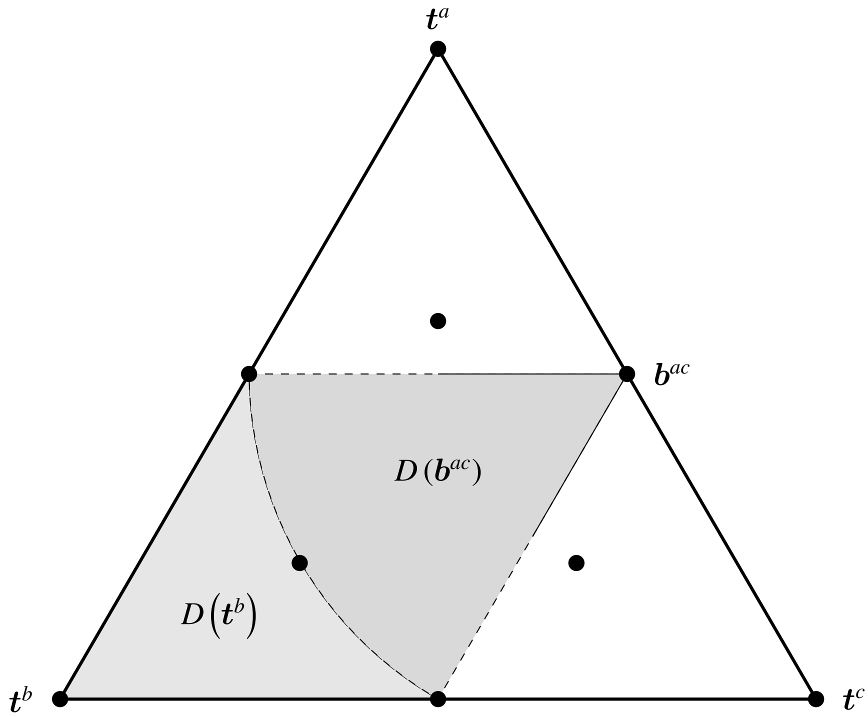

Figure 5: Strictly concave power: $\pi(C, x) = \sum_{i \in C} \sqrt{x_i}$

n, the unique symmetric stable set should be the set of allocations that split the allocation equally between a minimal majority of agents.

Third, solution concepts other than the stable set can be experimented with. For example, Jordan (2009) extended the stable set to the *legitimate set* in production pillage games. Chaturvedi (2016) has applied farsighted concepts from Chwe (1994) to pillage games. Duggan (2013) discussed other farsighted concepts, such as *uncovered sets*; in 3-AMPGs, the Gillies, Bordes and McKelvey uncovered sets are non-empty, and any stable set is a subset of the Gillies uncovered set.[6]

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[6]Proofs available on request. We are grateful to the AE for suggesting this.

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A Proofs

Proof of Theorem 3.

1. Together $B(a) = \emptyset$ and $K = \{\boldsymbol{t}^a\}$ imply that $v_b + v_c + 1 < v_a$. Therefore, $D(\boldsymbol{t}^a) = X \setminus \{\boldsymbol{t}^a\}$ and $S = \{\boldsymbol{t}^a\}$.

2. Note that $B(a) = \{\boldsymbol{t}^a\}$ implies $v_a + 1 = v_b + v_c$ and $\boldsymbol{t}^a = \boldsymbol{e}^a_b$. Moreover, $B(i) = \left\{\boldsymbol{t}^i\right\}$ if $v_i = v_a$, and $B(i) = \emptyset$ otherwise, for $i = b, c$. As $\boldsymbol{t}^i \notin K$ for $i = b, c$, it follows that $v_i + 1 < v_j + v_k$ for $i = b, c$, thus $B(i) = \emptyset$ for $i = b, c$. Let $\overset{\circ}{X} \equiv \{\boldsymbol{x} \in X \mid \boldsymbol{x} > \boldsymbol{0}\}$. Then, for any $\boldsymbol{x} \in \overset{\circ}{X}$, $\boldsymbol{x} \succ \boldsymbol{t}^i$ for $i = b, c$. Moreover, for any $\boldsymbol{x} \in \overset{\circ}{X}$, it follows that $v_i + x_i < v_j + v_k + (1 - x_i)$ for any $i, j, k \in I$, which implies that there exists $\boldsymbol{w}^{\boldsymbol{x}} \in X$ such that $w^{\boldsymbol{x}}_i = 0$ and $w^{\boldsymbol{x}}_j / w^{\boldsymbol{x}}_k = x_j / x_k$. Then, it follows that $\boldsymbol{w}^{\boldsymbol{x}} \succ \boldsymbol{x}$.

   Let $\bar{X}_{jk} \equiv \{\boldsymbol{x} \in X \mid x_i = 0\}$ for any $i, j, k \in I$. If $v_c + 1 < v_b$, then for any $\boldsymbol{w} \in \bar{X}_{bc} \setminus \left\{\boldsymbol{t}^b\right\}$, $\boldsymbol{t}^b \succ \boldsymbol{w}$. If $v_c + 1 \geq v_b$, then there exists $\boldsymbol{w}^{bc} \in \bar{X}_{bc}$ such that $v_b + w^{bc}_b = v_c + w^{bc}_c$. Then, for any $\boldsymbol{w} \in \bar{X}_{bc}$, $\boldsymbol{t}^b \succ \boldsymbol{w}$ if and only if $w_b > w^{bc}_b$; $\boldsymbol{t}^c \succ \boldsymbol{w}$ if and only if $w_c > w^{bc}_c$; and there exists $\boldsymbol{x}^{\boldsymbol{w}} \in \overset{\circ}{X}$ such that $\boldsymbol{x}^{\boldsymbol{w}} \succ \boldsymbol{w}$ for $\boldsymbol{w} = \boldsymbol{w}^{bc}$. In sum, cyclic relations of the dominance operator $\succ$ are observed over $\overset{\circ}{X}$ and $\bar{X}_{bc}$, which implies that none of allocations in $\overset{\circ}{X} \cup \bar{X}_{bc}$ can constitute a stable set. Applying the same argument to $\bar{X}_{ab} \setminus \{\boldsymbol{t}^a\}$ and $\bar{X}_{ac} \setminus \{\boldsymbol{t}^a\}$ proves that no stable set intersects with $X \setminus \{\boldsymbol{t}^a\}$.

   Hence, if a stable set exists, it must be that $S = \{\boldsymbol{t}^a\}$. However, $D(\boldsymbol{t}^a) \neq X \setminus \{\boldsymbol{t}^a\}$, as $v_a + x_a < v_b + v_c + (1 - x_a)$ for any $\boldsymbol{x} \in \overset{\circ}{X}$, which implies $\boldsymbol{t}^a \not\succ \boldsymbol{x}$. Therefore, $S \neq \{\boldsymbol{t}^a\}$, a contradiction. Hence, no stable set exists.

3. This implies that $-1 \leq -v_a + v_b + v_c < 1$, and so $\boldsymbol{e}^a_b \in X$. Suppose $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab}$.

Then, $v_c = 0$. Then, $\boldsymbol{t}^b \notin K$ implies $v_b + 1 < v_a$ and $v_b + v_c + 1 < v_a$, which contradicts $-1 \leq -v_a + v_b + v_c$. Thus, $\boldsymbol{e}^a_b \neq \boldsymbol{b}^{ab}$, which implies $v_c > 0$. Then, by Corollary 4, no stable set exists.

$\square$

Proof of Theorem 4. By $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$, Theorem 1 implies that $v_a + 1 \geq v_b + v_c$ and $v_b + 1 \geq v_a + v_c$, where the former implies $\beta^a_a \leq 1$. Moreover, the latter inequality implies $v_b + v_c + 1 \geq v_a$, which implies $\beta^a_a \geq 0$. Therefore, $B(a) \neq \emptyset$.

Consider $\boldsymbol{e}^a_b = \boldsymbol{t}^a$. Then, as $v_b + 1 \geq v_a + v_c$ by $\boldsymbol{t}^b \in K$ and $v_a + 1 = v_b + v_c$ by $\boldsymbol{e}^a_b = \boldsymbol{t}^a$, it follows that $v_a = v_b = v_c = 1$. Then, $v_c + 1 = v_a + v_b$, which contradicts $\boldsymbol{t}^c \notin K$ by Theorem 1. Thus, $\boldsymbol{e}^a_b \neq \boldsymbol{t}^a$ must hold, and $\beta^a_a < 1$.

Consider $\boldsymbol{e}^a_b \neq \boldsymbol{b}^{ab}$, which implies $v_c > 0$: by Corollary 4, no stable set exists.

Consider $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} \neq \boldsymbol{t}^b$, which implies $v_c = 0$. Then, it follows from $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} \neq \boldsymbol{t}^b$ and $v_c = 0$ that $\beta^a_a = \frac{1}{2}(1 + v_b - v_a) \in (0, 1)$, which implies $\boldsymbol{b}^{ab} \in K$ by Theorem 1. This contradicts $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$. Therefore, if $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab}$, then $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$.

Consider $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$. Then, as $\beta^a_a = 0$ when $v_c = 0$, $B(a) = \bar{X}_{bc} \equiv \{\boldsymbol{x} \in X \mid x_a = 0\}$. Further, for any $\boldsymbol{x} \in X \backslash B(a)$, $v_a + x_a > v_b + v_c + (1 - x_a)$ holds, which implies $\boldsymbol{t}^a \succ \boldsymbol{x}$.

As $v_a = 1 + v_b$ by $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$, either $v_a = 1$ and $v_b = 0$, or $v_a > 1$. The first case also implies that $v_a + v_b = 1 + v_c$, so that $\boldsymbol{t}^c \in K$ from Theorem 1; this contradicts $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$, which leaves only the second case, $v_a > 1$. This further implies $\boldsymbol{t}^a \succ \boldsymbol{t}^c = \boldsymbol{e}^a_c = (0, 0, 1)$. Then, by Corollary 3 and Lemma 5, we can conclude: $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$ if and only if $\bar{\boldsymbol{\beta}}^a \notin X$.

Note that $\bar{\boldsymbol{\beta}}^a \in X$ if and only if $v_b \leq 1$, since $\bar{\beta}^a_b = \frac{1 - v_b}{2}$. Thus, $\bar{\boldsymbol{\beta}}^a \notin X$ if and only if $v_b > 1$ (or equivalently, $v_a > 2$). When, $\bar{\boldsymbol{\beta}}^a \notin X$, Lemma 5 ensures that any allocation in $B(a) \backslash \left\{\boldsymbol{t}^b\right\}$ is dominated by $\boldsymbol{e}^a_b = \boldsymbol{t}^b$. Thus, $D\left(\boldsymbol{t}^a, \boldsymbol{t}^b\right) = X \backslash \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$. Finally, $\left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\} = K$ satisfies internal stability, so that $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$. In contrast, when $\bar{\boldsymbol{\beta}}^a \in X$, then Corollary 3 implies that whenever S exists, $\boldsymbol{t}^c = \boldsymbol{e}^a_c \in S$ holds. However, in this case, $\boldsymbol{t}^a \in S$ also holds by $\boldsymbol{t}^a \in K$, yielding a contradiction with $\boldsymbol{t}^a \succ \boldsymbol{t}^c = \boldsymbol{e}^a_c$. Thus, S does not exist when $\bar{\boldsymbol{\beta}}^a \in X$. $\square$

Proof of Theorem 5. By Theorem 1, $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$ implies that $v_c = 0$, so that $\boldsymbol{b}^{ab} = \boldsymbol{e}^a_b = \boldsymbol{e}^b_a$. Therefore, $B(a) \neq \emptyset$ and $B(b) \neq \emptyset$, where $B(a) = \left\{\boldsymbol{x} \in X \mid x_a = \bar{\beta}^a_a\right\}$ with $\bar{\beta}^a_a = \frac{v_b - v_a + 1}{2}$ and $B(b) = \left\{\boldsymbol{x} \in X \mid x_b = \bar{\beta}^b_b\right\}$ with $\bar{\beta}^b_b = \frac{v_a - v_b + 1}{2}$. Then, for any $\boldsymbol{x} \in X$ with $x_a > \bar{\beta}^a_a$, $v_a + x_a > v_b + v_c + (1 - x_a)$, which implies $\boldsymbol{t}^a \succ \boldsymbol{x}$. Likewise, for any $\boldsymbol{y} \in X$ with $y_b > \bar{\beta}^b_b$ we have $v_b + y_b > v_a + v_c + (1 - y_b)$, which implies $\boldsymbol{t}^b \succ \boldsymbol{y}$. Consider $\left\{\boldsymbol{x} \in X \mid x_a < \bar{\beta}^a_a, x_b < \bar{\beta}^b_b\right\}$. Then, for any $\boldsymbol{z} \in$

$\left\{\boldsymbol{x} \in X \mid x_a < \bar{\beta}^a_a, x_b < \bar{\beta}^b_b\right\}$,

$$v_a + z_a + v_b + z_b > v_c + (1 - z_a - z_b)$$

follows from $\boldsymbol{t}^c \notin K$ with Theorem 1. Thus, $\boldsymbol{b}^{ab} \succ \boldsymbol{z}$.

Let $\bar{\boldsymbol{\beta}}^a \in X$, so that $1 + v_a - 3v_b > 0$. In the case of $v_b > 0$, no stable set exists (by Corollary 4). Otherwise, when $v_b = 0$, $\boldsymbol{t}^b \in K$ implies that $1 \geq v_a$. However, $\boldsymbol{t}^c \notin K$ implies that $1 < v_a$, a contradiction. Therefore, $\bar{\boldsymbol{\beta}}^a \in X$ with $v_b = 0$ is impossible.

Let $\bar{\boldsymbol{\beta}}^a \notin X$. Then, $1 + v_a - 3v_b \leq 0$, so that $1 + v_b - 3v_a \leq 0$, and $\bar{\boldsymbol{\beta}}^b \notin X$. Then, by Lemma 5, for any $\boldsymbol{w} \in B(a)$, $\boldsymbol{e}^a_b \succ \boldsymbol{w}$ holds. Likewise, for any $\boldsymbol{w} \in B(b)$, $\boldsymbol{e}^b_a \succ \boldsymbol{w}$ holds. As $\boldsymbol{b}^{ab} = \boldsymbol{e}^a_b = \boldsymbol{e}^b_a$, $B(a) \cup B(b) \subseteq D\left(\boldsymbol{b}^{ab}\right)$ holds. In summary, $D(K) = X \backslash K$ holds for $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$. Thus, $S = K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$. □

Proof of Theorem 6. As $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$, Theorem 1 implies that $1 + v_a \geq v_b + v_c$, $1 + v_b \geq v_a + v_c$, and $1 + v_c \geq v_a + v_b$. Therefore, $\beta^i_i \leq 1$ holds for all $i \in \{a, b, c\}$. Moreover, $1 + v_c \geq v_a + v_b$ implies $1 + v_b + v_c \geq v_a$, and so $\beta^i_i \geq 0$ holds for all $i \in \{a, b, c\}$. Thus, $B(a) \neq \emptyset$, $B(b) \neq \emptyset$, and $B(c) \neq \emptyset$. Consider two cases:

1. $v_c = 0$. Then, $\boldsymbol{t}^b \in K$ implies implies $1 \geq v_a - v_b$. As $\boldsymbol{b}^{ab} \notin K$, Theorem 1 implies that $1 = v_a - v_b$. Thus, $1 + v_b = v_a$ and $1 \geq v_a + v_b$ together imply that $v_b = 0$ and $v_a = 1$. Then, $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab} = \boldsymbol{t}^b$ so that $B(a) = \{\boldsymbol{x} \in X \mid x_a = 0\}$. Moreover, $\bar{\boldsymbol{\beta}}^a = \left(0, \frac{1}{2}, \frac{1}{2}\right) = \boldsymbol{b}^{bc} = \boldsymbol{s}^{bc}$. Then, by Lemma 5, for any $\boldsymbol{w} \in B(a)$, if $\boldsymbol{w} \in \left(\boldsymbol{t}^b, \bar{\boldsymbol{\beta}}^a\right)$, $\boldsymbol{t}^b \succ \boldsymbol{w}$; if $\boldsymbol{w} \in \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{t}^c\right)$, $\boldsymbol{t}^c \succ \boldsymbol{w}$ holds, as $\boldsymbol{t}^c = \boldsymbol{e}^a_c$. Finally, for any $\boldsymbol{x} \in X \backslash B(a)$, $\boldsymbol{t}^a \succ \boldsymbol{x}$ holds. Thus, $X \backslash \left(K \cup \left\{\bar{\boldsymbol{\beta}}^a\right\}\right) \subseteq D(K)$. Moreover, it can be seen that $\boldsymbol{t}^i \not\succ \bar{\boldsymbol{\beta}}^a$ and $\bar{\boldsymbol{\beta}}^a \not\succ \boldsymbol{t}^i$ for all $i \in \{a, b, c\}$. Thus, the stable set must be given by $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \bar{\boldsymbol{\beta}}^a\right\}$. As $\bar{\boldsymbol{\beta}}^a = \boldsymbol{b}^{bc} = \boldsymbol{s}^{bc}$, $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{bc}\right\} = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{s}^{bc}\right\}$ holds.

2. $v_c > 0$. First consider $v_c < v_a$. Then, $1 + v_a > v_b + v_c$ follows from $1 + v_c \geq v_a + v_b$. Then, by Corollary 4, no stable set exists. Now consider $v_c = v_a$, the symmetric case. Satisfying $1 + v_a \geq v_b + v_c$, $1 + v_b \geq v_a + v_c$, and $1 + v_c \geq v_a + v_b$ therefore forces $v_a = v_b = v_c = 1$. Then, $\boldsymbol{e}^a_b = \boldsymbol{t}^a$, $\boldsymbol{e}^b_a = \boldsymbol{t}^b$, and $\boldsymbol{e}^c_a = \boldsymbol{t}^c$, so that $B(a) = \{\boldsymbol{t}^a\}$, $B(b) = \left\{\boldsymbol{t}^b\right\}$, and $B(c) = \{\boldsymbol{t}^c\}$. Then, for each interval $\left[\boldsymbol{t}^i, \boldsymbol{t}^j\right]$, any $\boldsymbol{w} \in \left(\boldsymbol{t}^i, \boldsymbol{s}^{ij}\right)$ implies $\boldsymbol{t}^i \succ \boldsymbol{w}$, while any $\boldsymbol{w} \in \left(\boldsymbol{s}^{ij}, \boldsymbol{t}^j\right)$ implies $\boldsymbol{t}^j \succ \boldsymbol{w}$, for any $i, j \in \{a, b, c\}$. Thus, $\bar{X}_{jk} \backslash \left\{\boldsymbol{s}^{jk}\right\} \subseteq D\left(\boldsymbol{t}^j, \boldsymbol{t}^k\right)$ holds for any $j, k \in \{a, b, c\}$.

   Moreover, we can show that $\mathring{X} \subseteq D\left(\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right)$. Let $\boldsymbol{x} \in \mathring{X}$ with $x_i \geq \frac{1}{2}$, so that $x_j, x_k < \frac{1}{2}$. Then, a pillage from $\boldsymbol{x}$ to $\boldsymbol{s}^{jk}$ implies that $W = \{j, k\}$ and

$L = \{i\}$. Then, we have $\frac{3}{2} \leq v_i + x_i < 2$ while $v_j + x_j + v_k + x_k > 2$, which implies $v_j + x_j + v_k + x_k > v_i + x_i$. Thus, $\boldsymbol{s}^{jk} \in\!\!- \boldsymbol{x}$ holds. Now let $\boldsymbol{x} \in \overset{\circ}{X}$ with $x_i < \frac{1}{2}$, which implies there exists, say k such that $x_k < \frac{1}{2}$. Then, a pillage from $\boldsymbol{x}$ to $\boldsymbol{s}^{ik}$ implies that $W = \{i, k\}$ and $L = \{j\}$. Again $\boldsymbol{s}^{ik} \in\!\!- \boldsymbol{x}$ holds. This implies that $\overset{\circ}{X} \subseteq D\left(\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right)$.

In summary, $X \backslash \left(K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}\right) \subseteq D\left(K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}\right)$. Moreover, no allocation in $K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$ dominates another, satisfying internal stability. Therefore, $S = K \cup \left\{\boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\} = K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$.

□

Proof of Theorem 7. By Theorem 1, $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$ implies that $v_c = 0$, $1 + v_a > v_b$, $1 + v_b > v_a$, and $1 \geq v_a + v_b$. Then, $\boldsymbol{b}^{ab} = \boldsymbol{e}^a_b = \boldsymbol{e}^b_a$. Moreover, $B(a) \neq \emptyset$, $B(b) \neq \emptyset$ and $B(c) \neq \emptyset$, where $B(a) = \left\{\boldsymbol{x} \in X \mid x_a = \bar{\beta}^a_a\right\}$ with $\bar{\beta}^a_a = \frac{v_b - v_a + 1}{2} \in (0, 1)$, $B(b) = \left\{\boldsymbol{x} \in X \mid x_b = \bar{\beta}^b_b\right\}$ with $\bar{\beta}^b_b = \frac{v_a - v_b + 1}{2} \in (0, 1)$, and $B(c) = \left\{\boldsymbol{x} \in X \mid x_c = \bar{\beta}^c_c\right\}$ with $\bar{\beta}^c_c = \frac{v_a + v_b + 1}{2} \in (0, 1]$.

Let $1 > v_a + v_b$. Then, by Corollary 4, no stable set exists whenever $v_b > 0$. Assume $v_b = 0$. Then, $v_a < 1$, which implies that $v_a - v_c < 1$, and so $\boldsymbol{b}^{ac} \in K$ by Theorem 1, a contradiction. Therefore, $v_b > 0$ must hold.

Let $1 = v_a + v_b$. If $v_b = 0$, then $1 = v_a + v_b$ implies $1 = v_a$ while $1 + v_b > v_a$ implies $1 > v_a$, a contradiction. Thus, $v_b > 0$. Thus, $v_a < 1$. Assume $v_a > \frac{1}{2} > v_b$. Then, $1 + v_a > 3v_b$ holds, so by Corollary 4(ii), no stable set exists.

Thus, let $v_a = \frac{1}{2} = v_b$. Then, $\bar{\beta}^a_a = \frac{1}{2} = \bar{\beta}^b_b$ and $\bar{\beta}^c_c = 1$. Thus, $B(a) = \left\{\boldsymbol{x} \in X \mid x_a = \frac{1}{2}\right\}$, $B(b) = \left\{\boldsymbol{x} \in X \mid x_b = \frac{1}{2}\right\}$, and $B(c) = \{\boldsymbol{t}^c\}$. As usual, for any $\boldsymbol{x} \in X$ with $x_a > \frac{1}{2}$, $v_a + x_a > v_b + v_c + (1 - x_a)$ holds, which implies $\boldsymbol{t}^a \in\!\!- \boldsymbol{x}$. Likewise, for any $\boldsymbol{y} \in X$ with $y_b > \frac{1}{2}$, $v_b + y_b > v_a + v_c + (1 - y_b)$ holds, which implies $\boldsymbol{t}^b \in\!\!- \boldsymbol{y}$. Consider $\left\{\boldsymbol{x} \in X \mid x_a < \frac{1}{2}, x_b < \frac{1}{2}, x_c < 1\right\}$. Then, for any $\boldsymbol{z} \in \left\{\boldsymbol{x} \in X \mid x_a < \frac{1}{2}, x_b < \frac{1}{2}, x_c < 1\right\}$,

$$v_a + z_a + v_b + z_b > v_c + z_c$$

follows from $B(c) = \{\boldsymbol{t}^c\}$. Thus, $\boldsymbol{b}^{ab} \in\!\!- \boldsymbol{z}$. In summary, $X \backslash K \subseteq D\left(\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right)$, which implies $S = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$. □

Proof of Theorem 8. By Theorem 1, $\boldsymbol{b}^{ab}, \boldsymbol{b}^{ac} \in K$ implies $v_b = 0 = v_c$; $\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c \in K$ then implies $1 > v_a$. If $v_a = 0$, then $\boldsymbol{b}^{bc} = \left(0, \frac{1}{2}, \frac{1}{2}\right) \in K$ follows from Theorem 1, which is a contradiction. Therefore, $v_a > 0$. Moreover, $\boldsymbol{b}^{ab} = \boldsymbol{e}^a_b = \boldsymbol{e}^b_a$, $\boldsymbol{e}^a_c = \boldsymbol{b}^{ac} = \boldsymbol{e}^c_a$, $B(a) \neq \emptyset$, $B(b) \neq \emptyset$ and $B(c) \neq \emptyset$, where $B(a) = \left\{\boldsymbol{x} \in X \mid x_a = \bar{\beta}^a_a\right\}$

with $\bar{\beta}^a_a = \frac{-v_a+1}{2} \in (0,1)$, $B(b) = \left\{\boldsymbol{x} \in X \mid x_b = \bar{\beta}^b_b\right\}$ with $\bar{\beta}^b_b = \frac{v_a+1}{2} \in (0,1)$, and $B(c) = \left\{\boldsymbol{x} \in X \mid x_c = \bar{\beta}^c_c\right\}$ with $\bar{\beta}^c_c = \frac{v_a+1}{2} \in (0,1)$.

Let $v_a < \frac{1}{3}$. Then, by Corollary 4(ii), no stable set exists.

Let $v_a \in \left[\frac{1}{3}, 1\right)$. Then, as usual, for any $\boldsymbol{x} \in X$ with $x_a > \frac{-v_a+1}{2}$, $\boldsymbol{t}^a \succ \boldsymbol{x}$. Likewise, for any $\boldsymbol{y} \in X$ with $y_b > \frac{v_a+1}{2}$, $\boldsymbol{t}^b \succ \boldsymbol{y}$, and for any $\boldsymbol{z} \in X$ with $z_c > \frac{v_a+1}{2}$, $\boldsymbol{t}^c \succ \boldsymbol{z}$. Consider $\left\{\boldsymbol{x} \in X \mid x_a < \frac{-v_a+1}{2}, x_b < \frac{v_a+1}{2}, x_c < \frac{v_a+1}{2}\right\}$. Then, for any $\boldsymbol{w} \in \left\{\boldsymbol{x} \in X \mid x_a < \frac{-v_a+1}{2}, x_b < \frac{v_a+1}{2}, x_c < \frac{v_a+1}{2}\right\}$,

$$v_a + z_a + v_b + z_b > v_c + z_c.$$

Thus, $\boldsymbol{b}^{ab} \succ \boldsymbol{w}$.

Note that $v_a \geq \frac{1}{3}$ implies that $\bar{\boldsymbol{\beta}}^i \in X$ if and only if $v_a = \frac{1}{3}$ for $i = b, c$. Moreover, in this case, $\bar{\boldsymbol{\beta}}^a = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\bar{\boldsymbol{\beta}}^b = \left(0, \frac{2}{3}, \frac{1}{3}\right) = \boldsymbol{e}^b_c$, and $\bar{\boldsymbol{\beta}}^c = \left(0, \frac{1}{3}, \frac{2}{3}\right) = \boldsymbol{e}^c_b$ hold when $v_a = \frac{1}{3}$.

Let $v_a = \frac{1}{3}$. Then, for any $\boldsymbol{y} \in \left(\boldsymbol{e}^b_a, \boldsymbol{e}^b_c\right)$, $\boldsymbol{b}^{ab} \succ \boldsymbol{y}$ follows from Lemma 5. Likewise, for any $\boldsymbol{z} \in \left(\boldsymbol{e}^c_a, \boldsymbol{e}^c_b\right)$, $\boldsymbol{b}^{ac} \succ \boldsymbol{z}$ holds, as do $\boldsymbol{t}^b \succ \boldsymbol{e}^b_c$ and $\boldsymbol{t}^c \succ \boldsymbol{e}^c_b$. In contrast, for any $\boldsymbol{x} \in \left(\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$, $\boldsymbol{b}^{ab} \succ \boldsymbol{x}$ holds, while for any $\boldsymbol{x} \in \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right)$, $\boldsymbol{b}^{ac} \succ \boldsymbol{x}$ holds, by Lemma 5. Thus, we have $X \backslash \left(K \cup \bar{\boldsymbol{\beta}}^a\right) \subseteq D(K)$. As $\bar{\boldsymbol{\beta}}^a$ neither dominates nor is dominated by any allocation in K, $K \cup \bar{\boldsymbol{\beta}}^a$ satisfies internal stability. Therefore, $S = K \cup \bar{\boldsymbol{\beta}}^a$.

Let $v_a \in \left(\frac{1}{3}, 1\right)$. Then, for any $\boldsymbol{y} \in \left(\boldsymbol{e}^b_a, \boldsymbol{e}^b_c\right]$, $\boldsymbol{b}^{ab} \succ \boldsymbol{y}$ holds, while, for any $\boldsymbol{z} \in \left(\boldsymbol{e}^c_a, \boldsymbol{e}^c_b\right]$, $\boldsymbol{b}^{ac} \succ \boldsymbol{z}$ holds by Lemma 5. Moreover, for any $\boldsymbol{x} \in \left(\boldsymbol{e}^a_b, \bar{\boldsymbol{\beta}}^a\right)$, $\boldsymbol{b}^{ab} \succ \boldsymbol{x}$ holds, while for any $\boldsymbol{x} \in \left(\bar{\boldsymbol{\beta}}^a, \boldsymbol{e}^a_c\right)$, $\boldsymbol{b}^{ac} \succ \boldsymbol{x}$ holds, by Lemma 5, where $\bar{\boldsymbol{\beta}}^a = \left(\frac{1-v_a}{2}, \frac{1+v_a}{4}, \frac{1+v_a}{4}\right)$.

To conclude, $X \backslash \left(K \cup \bar{\boldsymbol{\beta}}^a\right) \subseteq D(K)$ holds. As $\bar{\boldsymbol{\beta}}^a$ neither dominates nor is dominated by any allocation in K, $K \cup \bar{\boldsymbol{\beta}}^a$ satisfies internal stability. Therefore, $S = K \cup \bar{\boldsymbol{\beta}}^a$. □