KOCHI UNIVERSITY OF TECHNOLOGY

Social Design Engineering Series *SDES-2023-2*

Sraffian indeterminacy of steady-state equilibria in the Walrasian general equilibrium framework

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May 10, 2023

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April 7, 2023

Abstract

In contrast to Mandler's (1999a; Theorem 6) generic determinacy of steady-state equilibria, we first show that any non-trivial steady-state equilibrium is *indeterminate* under a general overlapping generation economy with a fixed Leontief technique. We also check that this indeterminacy is *generic*. These results are obtained by explicitly introducing a general model of every generation's utility function and individual optimization program to the overlapping generation economy, which also verifies that Mandler's (1999a; section 6) claim on generic determinacy is invalid. We also argue the distinctiveness of our results in comparison with the standard literature, like Calvo (1978), of overlapping generation indeterminacy.

JEL Classification Code: B51, D33, D50.

Keywords: Sraffian indeterminacy; functional income distribution; general equilibrium framework

*We are grateful to the two referees and an editor, Neri Salvadori, of this journal for their useful comments and suggestions. We are also specially thankful to Michael Mandler, Nobusumi Sagara, Roberto Veneziani, Kazuya Kamiya, Mamoru Kaneko, Gil Skillman, Peter Matthews, Kazuhiro Kurose, and Peter Skott for some useful discussions regarding this paper's subject. An earlier version of this paper was presented at Asian Meeting of Econometric Society 2019, ASSA 2018, and the Annual Meeting of Japanese Economic Association 2018. We are grateful to all the participants of the sessions where the paper was discussed, but in particular, to Enrico Bellino and Daniel Saros for their helpful comments and suggestions. We are also thankful to Rylan Chinnock for his advice on English. The usual disclaimer applies. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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1 Introduction

In classical and Marxian economics, functional income distribution is determined by means of various conditions, including not only market competition but also some historical and institutional conditions of the capitalist society as a whole. This view is supported by the analysis of Sraffa's (1960) system of price equations, which represents the long-period equilibrium under free competition and is known by its *one-degree of freedom* feature: the number of unknown variables is one greater than that of equations. This feature implies the underdetermination of price equations in that one of the wage and interest rates should be the parameter of the market mechanism, which must be determined outside of market competition in order to close the system of the equations.

Mandler (1999a) named this underdetermination feature '*Sraffian indeterminacy*'[1] and critically examined it within a Walrasian general equilibrium framework, where the Sraffian system of price equations is identical to zero-profit condition for a Walrasian competitive equilibrium. First, Mandler (1999a, section 3) confirms the generic indeterminacy of non-stationary equilibria within a two-stage sequential equilibrium framework with fixed production coefficients and the price inelastic supply of endowments, where the only unknown variables in the system of the second-period continuation equilibrium equations are the prices of commodities, the wage rate, and an interest rate. Moreover, the number of those variables is one greater than that of such equations. Second, Mandler (1999a, section 6) shows that steady-state equilibria are generically determinate, given overlapping generation economies with a fixed labor endowment and endogenous supplies of capital goods.[2] He then claims that his sequential equilibrium approach to Sraffian indeterminacy, developed in Mandler (1999a, section 3), is the sole possible way of defending Sraffa's idea of explaining income distribution by social and institutional conditions.

Fratini and Levrero (2011) criticize Mandler's sequential equilibrium approach to Sraffian indeterminacy. In a two-period intertemporal economy considered by Mandler (1995, 1999a; section 3), Fratini and Levrero (2011) see that the capital goods available at the beginning of the second period are produced in the first period as if there were Arrow-Debreu complete markets. Consequently, the quantities of capital goods that are produced in the first period and are delivered at the beginning of the second period will be those which would be realized in the Arrow–Debreu equilibrium.[3] In contrast, under the framework

[1] Here, the term 'Sraffian indeterminacy' refers to cases that market competition *alone* cannot determine equilibrium in the capitalist economy, in that one of the wage rate and the uniform rate of interests is a parameter of the market mechanism, and equilibrium can be represented by a continuous function of this parameter. It does not mean that the long-period equilibrium is indeterminate *under the framework of the Sraffian theory*, where various factors other than market competition are involved to explain the determination of that equilibrium.

[2] As Mandler (1999a, p. 699) himself points out, the Walrasian system of general equilibrium has an inherent problem of overdetermination: when the endowment of reproducible means of production is arbitrarily given, the system of equations is overdetermined under the uniform rate of profit. For further details of the implications of this issue, see Eatwell (1999), Garegnani (1990), and Petri (2004).

[3] Because of the latter property, the quantities of capital goods are necessary and sufficient

of two-period sequential trading considered by Mandler (1995, 1999a; section 3), the rental prices of those capital goods at the beginning of the second period can be arbitrarily determined without reference to the first-period prices of produced capital goods. That is the source of indeterminacy of equilibrium prices in the second period. In other words, the indeterminacy arises since the second-period rental prices could deviate from the expected prices, which are formed in the first period on the ground of the (counterfactual) Arrow-Debreu complete market equilibrium to guarantee the uniformity of effective rates of return.

In summary, Fratini and Levrero (2011) claim that Mandler's sequential indeterminacy is due to an *ad hoc* use of the tendency of the uniform rates of return of capital goods. Moreover, they argue that Mandler's analysis would bring us back to methodological questions about the Walrasian intertemporal general equilibrium theories and eventually justify Sraffa's rediscovery of the surplus approach of the classical economists and Marx to value and distribution.[4] In contrast, they provide no criticism against Mandler's another main argument: the *generic determinacy of steady-state equilibria*.

As a complement to Fratini and Levrero (2011), this paper develops an immanent criticism against Mandler's (1999a, section 6, Theorem 6) generic determinacy of steady-state equilibria. Mandler's Theorem 6 (1999a, section 6) is referred to verify the following claim in Mandler (1999b, p. 48): "if we incorporate preferences and demand into Sraffa's *long-run framework*, the neoclassical mechanism for determinacy will close the model: if w, r or goods prices were to deviate from an equilibrium configuration, the long-run demand for labour would change, violating market-clearing." Against this argument, we will construct a general model of overlapping generation economies and introduce the same definition of steady-state equilibrium as Mandler (1999a; section 6, p. 705), in which *a steady-state equilibrium is shown to be generically indeterminate*.

This result suggests that, unlike Mandler's claim, the lack of preferences and demands for goods and factors as well as of the market clearing condition would not be essential for the emergence of Sraffian indeterminacy, and introducing outside factors of market competition, such as some historical and institutional conditions of the capitalist society, should be indispensable for the determination of a steady-state equilibrium. Thus, the classical and Marxian views of the functional income distribution can be robust *even* if a Walrasian general equilibrium framework is applied. In other words, our analysis indicates that Sraffa's view of explaining income distribution by social and institutional conditions would capture a fundamental law in the capitalist economy, that should be established regardless of whether individuals are rational and have perfect

to produce an optimal level of commodity bundle with full employment of labor in the second period.

[4] The surplus approach views distribution as the result of social conditions that are more fundamental than those determining relative prices, rather than explaining it on the basis of the substitution principle among factors and goods.

foresight.[5]

In the rest of this paper, section 2 introduces a model of overlapping generation economies and defines the steady-state equilibrium, and then argues that Mandler's (1999a; section 6) claim on the generic determinacy of such an equilibrium fails to verify. Section 3 argues for the generic indeterminacy of such an equilibrium, contrary to Mandler's (1999a; section 6) claim. Finally, section 4 provides concluding remarks. The general existence theorem of an one-dimensional continuum of steady-state equilibria is provided in the Appendix.

2 An overlapping generation economy

As argued in section 1, Mandler (1999a, section 6) claims that generic determinacy is observed for steady-state equilibria in an overlapping generation economy with a simple Leontief production model, where he begins with an abstract Marshallian demand function of every generation as the primitive data for the overlapping generation (OLG hereafter) economy, and no explicit information about the underlying economy such as each generation's utility function and her optimization program is provided. Given this setting, he claims that in a long-run OLG setting the number of equilibrium equations and that of unknown variables are identical because none of the market-clearing equations are redundant by means of Walras' law (Mandler, 1999a; section 6; p. 704).

In this section, we will first examine Mandler's (1999a, section 6) claim, following his own model formulation and Walras' law. A simple OLG model is constructed, in which each generation $t = 1, 2, \ldots,$ lives for two periods. Each generation can earn only from labor supply in his youth but in his old age can earn both from labor supply and productive investment of his past saving. Let $\omega_l^b > 0$ (resp. $\omega_l^a \geqq 0$) be the labor endowment of one generation when he is young (resp. when he is old), and so $\omega_l \equiv \omega_l^b + \omega_l^a > 0$. Assume in the following that every generation has a common preference over his lifetime consumption activities, and labor is supplied inelastically for every generation in all of his ages.

There are $n \geqq 2$ commodities which are produced in this economy and used as consumption goods or capital goods, respectively. Let (A, L) be a *Leontief production technique* prevailing in this economy, where A is a $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients and L is a $1 \times n$ positive row vector of direct labor coefficients.

Let $z_b : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^n$ (*resp.* $z_a : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^n$) be a *Marshallian demand function* of every generation t in his youth (*resp.* in his old age) such that for each commodity price vectors $p_t, p_{t+1} \in \mathbb{R}_+^n$, each wage rates $w_t, w_{t+1} \in \mathbb{R}_+$, and a profit factor $1 + r_{t+1} \in \mathbb{R}_+$, $z_k(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) \in \mathbb{R}_+^n$ is a consumption vector purchasable for

[5] Needless to say, though our OLG model developed in this paper presumes that every generation is economic rational, it does not intend to claim that the Walrasian analysis is uniquely appropriate nor involve any criticism against the surplus approach.

every generation when his age is $k = b, a$. The demand function z_k is assumed to be *continuously differentiable* and satisfies *homogeneity*: for $k = b, a$,

$$\begin{aligned} z_k\left(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}\right) &= z_k\left(\lambda p_t, \lambda w_t, \lambda p_{t+1}, \lambda w_{t+1}, 1 + r_{t+1}\right) \\ &= z_k\left(p_t, w_t, \lambda p_{t+1}, \lambda w_{t+1}, \lambda\left(1 + r_{t+1}\right)\right) \end{aligned}$$

for any $\lambda > 0$ and every $\left(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}\right) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$; and *Walras' law*.

When z_k is evaluated at stationary prices $(p_t, w_t) = (p_{t+1}, w_{t+1}) = (p, w)$ for every t, we will use the notation $z_k(p, w, 1 + r)$ for $k = a, b$. Let $z(p, w, 1 + r) \equiv z_b(p, w, 1 + r) + z_a(p, w, 1 + r)$ be the aggregate demand function at every period t when the market prices are stationary. An *overlapping generation economy* is given by a profile $\left\langle (A, L); \left(\omega_l^b, \omega_l^a\right); (z_b, z_a)\right\rangle$ in Mandler (1999a; section 6, p. 705). Then, Mandler defines a steady-state equilibrium as follows:

Definition 1 [Mandler (1999a; section 6, Definition D6.2)]: A *steady-state equilibrium* under the overlapping generation economy $\left\langle (A, L); \left(\omega_l^b, \omega_l^a\right); (z_b, z_a)\right\rangle$ is a pair of a stationary price vector $(p, w, 1 + r) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and a gross output vector $y \in \mathbb{R}^n_+$, such that the following conditions hold:

$$p \leqq (1 + r) pA + wL, \qquad \text{(a)}$$

$$y \geqq z(p, w, 1 + r) + Ay, \qquad \text{(b)}$$

where $z(p, w, 1 + r) = z_b(p, w, 1 + r) + z_a(p, w, 1 + r)$; and

$$Ly \leqq \omega_l. \qquad \text{(c)}$$

Mandler (1999a; section 6, p. 705) assumes that $z(p, w, 1 + r) > \mathbf{0}$ for any $(p, w, 1 + r) > \mathbf{0}$. In this case, by the inequality (b) of Definition 1, the equilibrium output vector is positive: $y > \mathbf{0}$, whenever the equilibrium prices are positive: $(p, w, 1 + r) > \mathbf{0}$. Moreover, all of the above (a), (b), and (c) of Definition 1 hold with equality.

2.1 Invalidity of Mandler's (1999a, Theorem 6) claim on generic determinacy of the steady-state equilibrium

In the above system of equilibrium equations (a)-(c) with equality, there are $2n + 1$ equations while $2n + 1$ unknowns, given that one of the n commodities can be chosen as the *numeràire*. Mandler's claim of generic determinacy of the steady-state equilibrium is based on *his view that none of the* $2n + 1$ *equations can be redundant by means of Walras' law.* Note that **Mandler's (1999a; section 6, p. 704) own definition of Walras' law at stationary prices** is represented by:

$$(1 + r)\left(pz_b - w\omega_l^b\right) + pz_a - w\omega_l^a = 0. \qquad (\alpha)$$

To examine his claim, let $((p, w, 1 + r), y)$ be a solution of the equations (a) and (b) with equality. Let us multiply both sides of the equation (a) from the right by $y \geqq \mathbf{0}$, which implies (a') $py = (1+r)\,pAy + wLy$, while let us multiply both sides of the equation (b) from the left by $p \geqq \mathbf{0}$, which implies (b') $py = pz\,(p, w, 1+r) + pAy$. Then, from (a') and (b'), we have (b") $pz\,(p, w, 1+r) = rpAy + wLy$. On the other hand, Mandler's Walras' law (α) can be rewritten as $(pz\,(p, w, 1+r) - w\omega_l) + r\left(pz_b - w\omega_l^b\right) = 0$, which is further reduced to $(wLy - w\omega_l) + \left(rpAy + r\left(pz_b - w\omega_l^b\right)\right) = 0$ by substituting (b") to (α). From the last equation, the equation (c) $wLy - w\omega_l = 0$ automatically follows whenever all the savings of the young are financed to productive investments: $w\omega_l^b - pz_b = pAy$. Thus, equation (c) becomes redundant, so the above-mentioned Mandler's view does not hold.

Thus, for Mandler's claim of generic determinacy to hold, it must be possible to have $w\omega_l^b - pz_b > pAy$ in the steady-state equilibrium. In this respect, some of the literature on OLG models discuss that a portion of the savings would be devoted to *non-productive investments*.[6]

However, the data of the Marshallian demand functions in the underlying economy $\left\langle (A, L)\,; \left(\omega_l^b, \omega_l^a\right); (z_b, z_a)\right\rangle$ might be too abstract to identify whether an underlying individual optimization can achieve $w\omega_l^b - pz_b = pAy$ or not. Moreover, equation (c) can be redundant in the steady-state equilibrium even if a portion of the savings can be devoted to non-productive investments, unless the rate of return to such investments is *at least as equal as* that of productive investments.[7] In this respect, Mandler's Walras' law (α) specifies that the rate of return to the young's savings $\left(w\omega_l^b - pz_b\right)$ is always equal to $(1+r)$ regardless of how and where the savings are devoted to, which implies that both of non-productive and productive investments are equally efficient even when prices are stationary. However, without explicitly showing what kinds of non-productive investments are available and equally efficient to productive investments, Mandler's (1999a, Theorem 6) claim on generic determinacy of the steady-state equilibrium cannot be verified.

2.2 An explicit model of all generations' utility function and individual optimization program

In contrast to Mandler's (1999a, section 6) approach, we introduce, in the following argument, an explicit model of all generations' utility function and individual optimization program, where individual optimal actions would finance non-productive investments if the return of productive investments is not higher than that of non-productive investments. More precisely speaking, we model *speculative activities* to address the issues of non-productive investments.

Let $u : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ be a utility function of lifetime consumption activities, which is common to all generations. As usual, u is assumed to be continuous, strongly monotonic, and quasi-concave in each age's consumption space. In the whole of the following discussions, let $\omega_l^a = 0$ and so $\omega_l = \omega_l^b$ to refuse

[6] For instance, Genekoplos and Polemarchakis (2006) construct an OLG production economy, in which each agent can benefit from *holding money,* and so the benefit from holding money and the return from productive investments are indifferent in equilibrium.

[7] We will observe this point in detail later in section 3.

unessential complication. Thus, an *overlapping generation economy* is given by a profile $\langle (A, L); \omega_l; u \rangle$.

For each period t, let $p_t \in \mathbb{R}^n_+$ represent a vector of *prices* of n commodities prevailing at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailing at the end of this period; and $r_t \in [-1, \infty)$ represent an *interest rate* prevailing at the end of this period. Assume also, for each generation t, that $l^t \in \mathbb{R}_+$ represents t's labor supplied at the beginning of their youth; $\omega^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle for the purpose of saving monetary value $p_t\omega^{t+1}$, which will be chosen by generation t at the end of their youth and will be used in their old age; $\delta^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle purchased for the purpose of speculative activities by generation t at the beginning of their old age; $y^{t+1} \in \mathbb{R}^n_+$ represents a production activity vector decided by generation t at the beginning of their old age; z^t_b is the consumption bundle consumed by the generation t in their youth; and z^t_a is the consumption bundle consumed by generation t in their old age.

Each generation t in their youth is faced with the following optimization program MP^t: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max_{l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a} u\left(z^t_b, z^t_a\right)$$

subject to

$$\begin{aligned} p_t z^t_b + p_t \omega^{t+1} &\leqq w_t l^t, \\ l^t &\leqq \omega_l, \\ p_t \delta^{t+1} + p_t A y^{t+1} &= p_t \omega^{t+1}, \text{ and} \\ p_{t+1} z^t_a &\leqq p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

That is, each generation t can supply l^t amount of labor in her youth as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of her youth, she can save $p_t\omega^{t+1}$ amount of money and can purchase a consumption bundle z^t_b. By using the saved money $p_t\omega^{t+1}$, generation t at the beginning of her old age can purchase δ^{t+1} for speculative purposes and can purchase a vector of capital goods Ay^{t+1} as a productive investment. As an industrial capitalist, she can employ Ly^{t+1} amount of generation $t+1$'s labor. Then, at the end of her old age, she can earn $p_{t+1}\delta^{t+1}$ as the revenue of the speculative investment and can earn $p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}$ as the return on the productive investment. From these revenues, she can purchase a consumption bundle z^t_a.

Let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a\right)$ be a solution to the optimization program MP^t for each generation t. At the optimum, all of the weak inequalities in the above constraints should hold with equality, given the assumption of u. That is,

$$\begin{aligned} p_t z^t_b + p_t \omega^{t+1} &= w_t l^t, \\ l^t &= \omega_l, \text{ and} \\ p_{t+1} z^t_a &= p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

Note that the production activity vector y^{t+1}, planned by generation t at the beginning of old age, should satisfy the profit maximization condition. As market prices should satisfy the zero-profit condition in equilibrium, the following condition holds for every period $t+1$, where $t \geq 0$:

$$p_{t+1} \leqq (1 + r_{t+1})\, p_t A + w_{t+1} L.$$

Therefore, the profit maximization condition in equilibrium for every period $t+1$ is represented by:

$$p_{t+1} y^{t+1} = (1 + r_{t+1})\, p_t A y^{t+1} + w_{t+1} L y^{t+1}.$$

Thus, the revenue constraint $p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}$ of generation t at the end of the old age can be reduced to

$$p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + (1 + r_{t+1})\, p_t A y^{t+1}.$$

Given a sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}))$ be the Marshallian demand vectors of each generation $t = 1, 2, \ldots$, derived from a solution to the problem MP^t of utility maximization under the budget constraint. Then, a competitive equilibrium can be formulated as follows.

Definition 2: A *competitive equilibrium* under the overlapping generation economy $\langle (A, L); \omega_l; u \rangle$ is a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ and sequence of each generation's optimal actions $\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)\right\}_{t \geq 1}$ satisfying the following conditions:

$$p_t \leqq (1 + r_t)\, p_{t-1} A + w_t L \ \ (\forall t), \tag{1.1}$$

$$\delta^t + y^t \geqq z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + \omega^{t+1} \ \ (\forall t), \tag{1.2}$$

where $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ is the aggregate consumption demands at each t;

$$\delta^t + A y^t \leqq \omega^t \ \ (\forall t), \tag{1.3}$$

$$\text{and } L y^t \leqq \omega_l^t \ \ (\forall t). \tag{1.4}$$

In the above definition, the excess demand condition in commodity markets is given by (1.2). In each period t, the aggregate consumption demand vector is given by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$. It may contain some zero components. For commodity i such that $z_i^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = 0$, it follows that in equilibrium, $\delta_i^t + y_i^t \geqq \omega_i^{t+1}$. In the inequality of excess demand condition (1.2) above, y^t is the gross output vector which is planned by generation $t-1$ at the beginning of period t and is harvested at the end of this period, while δ^t is the commodity bundle purchased by generation $t-1$ at the beginning of period t and is sold by generation $t-1$ at the end of period t.

In each period t, the capital market equilibrium condition is given by (1.3) of Definition 2. Note that the choice between the speculative investment δ^t and the productive investment Ay^t is made by generation $t-1$ at the beginning of the old age. Moreover, the bundle of saving commodities ω^t is chosen by generation $t-1$ at the end of the young age.

In each period t, the labor market equilibrium condition is given by (1.4) of Definition 2. Note that the aggregate labor demand Ly^t is chosen by generation $t-1$ in their old age, while the aggregate labor supply ω_l^t is given by generation t at the young age.

Now, a steady-state equilibrium is defined as a specific case of competitive equilibria given in Definition 2.

Definition 1*: A *steady-state equilibrium* under the overlapping generation economy $\langle (A, L); \omega_l; u \rangle$ is a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ associated with

$$\left\{ \left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t (\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t (\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \right) \right\}_{t \geq 1},$$

such that there exists a profile of a stationary price vector (p, w, r), a gross output vector $y \geq \mathbf{0}$, and a speculative activity vector $\delta \geqq \mathbf{0}$, satisfying $(p_t, w_t, r_t) = (p, w, r)$, $y^{t+1} = y$, $\delta^{t+1} = \delta$, $\omega^{t+1} = Ay + \delta$, $z_b^t (\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b (p, w, r)$, and $z_a^t (\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a (p, w, r)$ for every t, and the inequalities (a), (b), and (c) in Definition 1 hold true.

In particular, a steady-state equilibrium $((p, w, r), y)$ is called *non-trivial* if $z (p, w, r) \geq \mathbf{0}$, where $z (p, w, r) \equiv z_b (p, w, r) + z_a (p, w, r)$.

Note that if a steady-state equilibrium (p, w, r) is non-trivial, then its associated equilibrium production activities are $y > \mathbf{0}$ by the inequality (b) and the indecomposability of A. By using this fact, it can be shown that all of (a), (b), and (c) in Definition 1 hold in equality for non-trivial steady-state equilibria.

In the above definitions, the choice between speculative investment δ^{t+1} and productive investment Ay^{t+1} is the consequence of each generation's optimal action in the program MP^t. Therefore, $\delta = \mathbf{0}$ can be optimal under the steady-state equilibrium whenever the equilibrium interest rate r is non-negative.

To see the last point, examine under what conditions the market equilibrium holds with no speculative activity, $\delta^{t+1} = \mathbf{0}$ $(\forall t)$. Note that if the whole monetary wealth $p_t \omega^{t+1}$ of generation t is devoted to productive investments, she would earn $(1 + r_{t+1}) p_t \omega^{t+1}$, while if it is used for speculative investments, she would earn $p_{t+1} \omega^{t+1}$. Therefore, allocating her whole monetary wealth to productive investments is an optimal action for generation t at the beginning of her old age if and only if $(1 + r_{t+1}) p_t \omega^{t+1} \geqq p_{t+1} \omega^{t+1}$. In general, if

$$(1 + r_{t+1}) p_t \geqq p_{t+1}$$

holds for every period $t \geq 0$, then $\delta^{t+1} = \mathbf{0}$ is an optimal action for every generation t at the beginning of the old age. This inequality condition holds automatically under a stationary price system associated with $r \geqq 0$, as $(1 + r) p \geqq p$ holds for $r \geqq 0$.

In contrast, under a stationary price system associated with $r < 0$, every generation would devote all of her wealth to speculative investments. Then, no production takes place in every period, and so no consumption good can be supplied in every period. Thus, if a steady-state equilibrium is associated with $r < 0$, it would be only a trivial one. As we are interested in the non-trivial case of equilibria, we will focus on the case with $r \geqq 0$ in the following argument.

3 Indeterminacy of non-trivial steady-state equilibria

In this section, we show that a non-trivial steady-state equilibrium is generically indeterminate. Firstly, again following Mandler (1999a), let us formulate the notion of indeterminacy in this model.

Definition 3: Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, a non-trivial steady-state equilibrium $((p, w, r), y)$ under this economy is *indeterminate* if for any $\varepsilon > 0$, there is another non-trivial steady-state equilibrium $((p', w', r'), y')$ such that $(p', w', r') \neq (p, w, r)$ and $\|(p', w', r') - (p, w, r)\| < \varepsilon$.

It should be worth emphasizing that indeterminacy of a non-trivial steady-state equilibrium requires a continuum of non-trivial steady-state equilibria including this particular one. Such a continuum could be represented by (a part of) the *wage-interest rate curve* derived from the Leontief technique (A, L).

Let the profile $((p, w, r), y)$ be a non-trivial steady-state equilibrium. It can be shown that it is indeterminate. To see this point, let us examine the system of equations that characterizes the non-trivial steady-state equilibrium, which is given as follows:

$$p = (1 + r) pA + wL; \tag{1}$$
$$y = z(p, w, r) + Ay; \text{ and} \tag{2}$$
$$Ly = \omega_l. \tag{3}$$

Note that (1) has n equations, (2) has n equations, and (3) has one equation. In contrast, there are n unknown variables regarding the vector y and there are $(n - 1) + 2$ unknown variables regarding (p, w, r), assuming hereafter that commodity n is selected as the *numeràire*. Thus, there are $2n + 1$ unknown variables in the system of $2n+1$ equations. However, we can decrease the number of equations using Walras' law. Based on this, we can show the indeterminacy of the non-trivial steady-state equilibrium in terms of Definition 3.

Given a non-trivial steady-state equilibrium $((p, w, r), y)$, define $\bar{p} \equiv (\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, 1)$ and the associated system of equilibrium equations as follows:

$$F(\bar{p}, w, r, y) \equiv \begin{bmatrix} z(p, w, r) - [I - A]\, y \\ (\bar{p} - (1 + r)\bar{p}A - wL)^T \end{bmatrix}.$$

By the definition of non-trivial steady state-equilibrium, $F(\bar{p}, w, r, y) = \mathbf{0}$ holds. Note that the mapping F does not contain the counterpart of equation (3). This is because the equation (3) is shown to be redundant, as discussed below in the proof of Theorem 1. Therefore, let us introduce the notion of regular equilibria by means of this F.

Definition 4 (Mandler (1999a)): Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, a non-trivial steady-state equilibrium $((p, w, r), y)$ under this economy is *regular* if the Jacobian of $F(\bar{p}, w, r, y) = \mathbf{0}$ has full row rank.

Now, we are ready to argue the indeterminacy of non-trivial steady-state equilibria, which is summarized as follows:

Theorem 1: Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above, and let $((p, w, r), y)$ be a non-trivial steady-state equilibrium under this economy. Then, it is *indeterminate*.

Proof. First, let us show that the equation (3) is redundant by means of Walras' law. In the overlapping generation economy, Walras' law is generally given by the following equation:

$$\left[p_t \left(z_b^t + z_a^{t-1}\right) + p_t \omega^{t+1}\right] - \left[p_t \delta^t + (1 + r_t) p_{t-1} A y_t + w_t \omega_l^t\right] = 0, \tag{4}$$

which is derived from the aggregation of $p_t z_b^t + p_t \omega^{t+1} - w_t \omega_l^t = 0$ and $p_t z_a^{t-1} - p_t \delta^t - (1 + r_t) p_{t-1} A y^t = 0$. Moreover, (4) can be reduced to the following form under stationary prices:

$$\left[p \left(z_b^t + z_a^{t-1}\right) + p \omega^{t+1}\right] - \left[p \delta^t + (1 + r) p A y_t + w \omega_l^t\right] = 0. \tag{4a}$$

Note that (4a) can be rewritten to the following form:

$$\left[p \left(z_b^t + z_a^{t-1}\right) + p A y_{t+1} + p \delta^{t+1}\right] - \left[p \delta^t + (1 + r) p A y_t + w \omega_l^t\right] = 0. \tag{4b}$$

As $z_b^t = z_b$, $z_a^{t-1} = z_a$, and $y_{t+1} = y_t = y$ hold for every t under the steady-state, (4b) can be reduced to

$$\left[p \left(z_b + z_a\right) + p \delta^{t+1}\right] - \left[p \delta^t + r p A y + w \omega_l\right] = 0. \tag{4b*}$$

Furthermore, $\delta_{t+1} = \delta_t = \delta$ also holds for every t under the steady-state. Indeed, $\omega^{t+1} = \omega^t = \omega$ holds in the steady-state. Thus, as $\delta_t + A y_t = \omega$ holds for every t whenever $p > \mathbf{0}$, $y_{t+1} = y_t = y$ implies $\delta_{t+1} = \delta_t = \delta$. Finally, $p > \mathbf{0}$ follows from the definition of Sraffian steady-state equilibrium prices (1), given the assumption of productive and indecomposable A and the positivity of L. Thus, (4b*) can be reduced to

$$p \left(z_b + z_a\right) - \left[r p A y + w \omega_l\right] = 0. \tag{4c}$$

Let us take a profile $((p, w, r), y)$ satisfying the system of equations (1) and (2). From (2), we have

$$\begin{gathered} p y = p z (p, w, r) + p A y \\ \text{where } z (p, w, r) = z_b (p, w, r) + z_a (p, w, r). \end{gathered} \tag{5}$$

By combining (1), (5) can be written as:

$$pz(p, w, r) = p(I - A)y = rpAy + wLy. \tag{5a}$$

Note that the profile $((p, w, r), y)$ meets Walras' law (4c), which implies that

$$pz(p, w, r) = rpAy + w\omega_l. \tag{6}$$

From (5a) and (6), we obtain the equation (3):

$$Ly = \omega_l.$$

Thus, the system of $2n + 1$ equations (1), (2), and (3) characterizing the non-trivial steady-state equilibrium $((p, w, r), y)$ can be reduced to the system of $2n$ equations (1) and (2), given the reduced form of Walras' law (4c). Then, since the system of $2n$ equations has $2n + 1$ unknown variables, it has freedom of degree one.

It can be shown that the equilibrium $((p, w, r), y)$ is regular as developed in the proof of Theorem 2 below. Then, the Jacobian matrix of the system of equations (1) and (2) at $((\overline{p}, w, r), y)$ has rank $2n$. Therefore, we can show the indeterminacy of the non-trivial steady-state equilibrium by applying the implicit function theorem. (A detailed proof is given in Theorem A2 of Appendix.) ■

As mentioned in section 2.1, given the same definition of steady-state equilibrium as Definition 1* in this paper, Mandler (1999a; section 6) claims that such an equilibrium is determinate, which is incompatible with Theorem 1. He reaches this conclusion by defining Walras' law as equation (α) and developing the following reasoning: "Due to the way in which $1+r$ appears in Walras' law, the standard argument that one of the equilibrium conditions is redundant is not valid in the present model" (Mandler, 1999a; section 6; p. 705). However, Mandler (1999a) has never proved that this statement is valid.

As we examined in section 2.1, given equation (α) as the representation of Walras' law, the validity of his claim of determinacy relies on the possibility that a portion of the young's savings will be devoted to non-productive investments. In contrast, in the above proof of Theorem 1, Walras' law is derived from a general model of individual optimization, which is always represented by equation (4c) independent of whether a portion of the young's savings is devoted to speculative investments or not. Note that equation (α) is equivalent to (4c) when $r = 0$. Therefore, the proof of Theorem 1 suggests that whenever $r = 0$, equation (α) can make one of the equilibrium equations redundant, and thus Mandler's claim on determinacy cannot hold. If $r > 0$, then all of the young's savings are devoted to productive investments as discussed in section 2.2, thus equation (α) is again equivalent to (4c), which implies that his claim of determinacy fails to hold in the class of economies specified in section 2.2.

3.1 Openness and genericity

Next, we examine the openness and genericity of parameter set of economies in which every non-trivial steady-state equilibrium is regular. The openness and genericity are related to the stability and coverage of indeterminacy in the perturbation of parameters characterizing the set of economies.

For the demand function of two generations z_b, z_a, labor endowment ω_ℓ and for $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function with similar form to Mandler (1999a) as

$$z_i(h) \equiv z_i^b(h) + z_i^a(h)$$

where

$$z_i^b(h) \equiv z_{bi}(p, w, r) + \frac{w}{p_i}h_i, \;\; z_i^a(h) \equiv z_{ai}(p, w, r) + \frac{w}{p_i}h^o$$

for each $i = 1, 2, ..., n$.

In order to preserve Walras' law and homogeneity, the perturbation of labor endowment is given as $\omega_l(h) \equiv \omega_\ell + \sum_{i=1}^n h_i + \frac{nh^o}{1+r}$.

Now define a function F on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$, n quantity variables $(y_1, y_2, ...y_n)$, and adding the parameter set (A, L, h) to $\mathbb{R}^{2n}$, *i.e.*

$$F : \mathbb{R}_{++}^{n-1} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{++}^{n} \times \mathbb{R}_{+}^{n^2} \times \mathbb{R}_{++}^{n} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n}$$

such that

$$F(\bar{p}, w, r, y, A, L, h) = \begin{bmatrix} z(h) - [I - A]\, y \\ (\bar{p} - (1+r)\bar{p}A - wL)^T \end{bmatrix}.$$

Definition 6: An *economy* is a profile of (A, L, h) where (A, L) is a Leontief production technique, in which A is $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients, L is $1 \times n$ positive row vector of direct labor coefficients, and $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$ is for perturbation.

An economy (A, L, h) is *regular* if every non-trivial steady-state equilibrium $((p, w, r), y)$ is regular, that is, the Jacobian $\boldsymbol{D}F$ has full-rank at $(\bar{p}, w, r, y)$. Denote the set of economies as P and the set of regular economies as P_R.

Theorem 2: P_R is open and has full measure in P.

Proof. Before examining whether P_R has full measure, let's first check whether the Jacobian $\boldsymbol{D}F$ has full rank with respect to $p_1, ...p_{n-1}, w, r, y_1, ..., y_n$ in order to check the regularity of an equilibrium in the economy (A, L, h). The system of equations above has $2n$ equations and $n+1$ price variables $(p_1, ..., p_{n-1}, w, r)$. Hence, the quantity variables $(y_1, ..., y_n)$ are to be determined simultaneously in the Jacobian. Including perturbed parameters, for any (A, L, h), $\boldsymbol{D}_{(y,\bar{p},w,r)}(F_{A,L,h}(\bar{p}, w, r, y))$ is given by:

$$\begin{bmatrix} [A - I] & \boldsymbol{D}_{\bar{p}} z(h) & \boldsymbol{D}_w z(h) & \boldsymbol{D}_r z(p, w, r) \\ \mathbf{0} & I_{n-1}^* - (1+r)A_{-n}^T & -L^T & -(\bar{p}A)^T \end{bmatrix}$$

where

$$\boldsymbol{D}_{\bar{p}}z(h)=\boldsymbol{D}_{\bar{p}}z(\bar{p},w,r)-\begin{bmatrix} \frac{w}{p_1^2}(h_1+h^o) & \mathbf{0} & \cdots & \\ \mathbf{0} & \frac{w}{p_2^2}(h_2+h^o) & \mathbf{0} & \cdots \\ & & \ddots & \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{w}{p_{n-1}^2}(h_{n-1}+h^o) \\ \mathbf{0} & \cdots & & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{D}_w z(h)=\boldsymbol{D}_w z(\bar{p},w,r)+[\frac{1}{p_1}(h_1+h^o),\frac{1}{p_2}(h_2+h^o),...,\frac{1}{p_{n-1}}(h_{n-1}+h^o),(h_n+h^o)]^T,$$

A^T is the transpose of A and A^T_{-n} is the $n\times(n-1)$ matrix obtained by deleting the n-th column of A^T, and

$$I^*_{n-1}=\begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix}.$$

Here, note that the last row of $\boldsymbol{D}_{\bar{p}}z(h)$ is nonzero because the last row of $\boldsymbol{D}_{\bar{p}}z(p,w,r)$ is non-zero. As we observed in the calculation result above, the Jacobian has full rank of $2n$ if the vectors $\begin{bmatrix} \mathbf{0} & I^*_{n-1}-(1+r)A^T_{-n} & -L^T & -(\bar{p}A)^T \end{bmatrix}$ are linearly independent. Note that it can be verified when the n column vectors $\begin{bmatrix} I^*_{n-1}-(1+r)A^T_{-n} & -L^T \end{bmatrix}$ are linear independent.

To see this, suppose on the contrary that $\begin{bmatrix} I^*_{n-1}-(1+r)A^T_{-n} & -L^T \end{bmatrix}$ are linear dependent. Then, there exists $\boldsymbol{\alpha}\in\mathbb{R}^{n-1}\setminus\{\mathbf{0}\}$ such that $\left[I^*_{n-1}-(1+r)A^T_{-n}\right]\boldsymbol{\alpha}^T=L^T$.[8] Let A^T_{n-1} be the $(n-1)\times(n-1)$ submatrix of A^T obtained by deleting the n-th row of A^T_{-n}, while L^T_{-n} be the $(n-1)\times 1$ column vector obtained by eliminating the n-th component of L^T. As $I-(1+r)A^T$ satisfies the Hawkins-Simon condition, it follows that $\left[I_{n-1}-(1+r)A^T_{n-1}\right]^{-1}\geq\mathbf{0}$, which implies that $\boldsymbol{\alpha}^T=\left[I_{n-1}-(1+r)A^T_{n-1}\right]^{-1}L^T_{-n}>\mathbf{0}$. However, as the n-the equation of $\left[I^*_{n-1}-(1+r)A^T_{-n}\right]\boldsymbol{\alpha}^T=L^T$ is given by $-\sum_{i=1}^{n-1}\alpha_i(1+r)a_{ni}=L_n$, which contradicts $\alpha_i>0$ for any $i=1,\ldots,n-1$. Therefore, the n column vectors $\begin{bmatrix} I^*_{n-1}-(1+r)A^T_{-n} & -L^T \end{bmatrix}$ are linear independent.

The full-measure claim of P_R is proven by the transversality theorem. Let's consider the perturbation of parameters (A,L,h) in $\mathbb{R}^{n^2}_+\times\mathbb{R}^n_{++}\times\mathbb{R}^{n+1}$. If 0 is a regular value of F at $(\bar{p},w,r,y)$ and $\boldsymbol{D}F$ has full rank $2n$ with respect to (A,L,h) in $\mathbb{R}^{n^2}_+\times\mathbb{R}^n_{++}\times\mathbb{R}^{n+1}$, then except for a set of $(A',L',h')\in\mathbb{R}^{n^2}_+\times\mathbb{R}^n_{++}\times\mathbb{R}^n$ of measure zero, $F_{A,L,h}(\bar{p},w,r,y):\mathbb{R}^{n-1}_{++}\times\mathbb{R}_{++}\times\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^{2n}$ has 0 as a regular value.

Define the Jacobian $\boldsymbol{D}F$ with respect to (A,L,h), which is denoted by $\boldsymbol{D}_{A,L,h}F$, as below:

[8] Note, as the $n-1$ column vectors $\left[I^*_{n-1}-(1+r)A^T_{-n}\right]$ are linear independent, the n-th column vector $-L^T$ must be a linear comibination of $\left[I^*_{n-1}-(1+r)A^T_{-n}\right]$ by the supposition.

$$
\boldsymbol{D}_{A,L,h}F = \begin{bmatrix}
\frac{w}{p_1} & & & 0 & \frac{w}{p_1} & y^T & & \\
& \ddots & & & & & & \mathbf{0} \\
& & \frac{w}{p_{n-1}} & 0 & \frac{w}{p_{n-1}} & & \ddots & \\
& 0 & & w & w & & y^T & \\
& & & & & & & \\
& & & & \mathbf{0} & & (*) & -wI_n \\
\end{bmatrix}
$$

where the row vector y^T is the transpose of y, I_n is the $n \times n$ identity matrix, $(*) = -(1+r)[p_1 I_n \ldots p_{n-1} I_n \; I_n]$ is $n \times n^2$ matrix. Here, each $p_i I_n$ is an $n \times n$ matrix:

$$
p_i I_n = \begin{bmatrix} p_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p_i \end{bmatrix}.
$$

The first $n+1$ columns are for $(h_1, ..., h_n, h^o)$, the next n^2 columns are for the components of A and the last n columns are for the components of L. We can see that the above matrix has full-rank.

As for openness, consider the contrary case. Suppose P_R is not open. Then there exists a sequence $\{(A, L, h)_k\}$ of non-regular economies converging to a regular economy $(A, L, h)_* \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $\{(\bar{p}, r, w, y)_k\}$ which converges to a regular equilibrium $(\bar{p}, r, w, y)_*$ at $(A, L, h)_*$. Then the corresponding Jacobian matrices $\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k$ of $2n$ rows and $2n+1$ columns exist, which have less than full rank. For a Jacobian matrix, we can pick $2n+1$ separate square submatrices of order $2n$. The determinants of square submatrices of order $2n$ are all zero. Now we can define a continuous function, say c, from the set of Jacobian matrices to the set of $2n+1$-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F_{A,L,h}$. Since $c(\boldsymbol{D}F_{A,L,h}) = (0, ..., 0) \in \mathbb{R}^{2n+1}$ for any $\boldsymbol{D}F_{A,L,h}$ of less than full rank, $c(\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k) = (0, ..., 0)_k \to (0, ..., 0) \in \mathbb{R}^{2n+1}$ as $k \to \infty$.

Since $\{(0, ..., 0)_k\}$ converging to $(0, ..., 0)$ is closed in $\mathbb{R}^{2n+1}$ and c is continuous, the inverse image $c^{-1}(\{(0, ..., 0)_k\}) = \left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\right\}$ is closed. Its elements are Jacobian matrices from $P \backslash P_R$ of less than full rank. Since $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\right\}$ is closed, $\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y)_*$ is contained in $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\right\}$.

Note that $c(\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y)_*) = (0, ..., 0) \in \mathbb{R}^{2n+1}$. This implies that the converging point of the sequence $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k\right\}$, each element of which is correspondingly defined from $(A, L, h)_k \in P \backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must also be non-regular. This contradicts our initial assumption. Therefore, the set of regular economies P_R is open. ∎

4 Concluding Remarks

In the above sections, we have argued that, under overlapping generation production economies with a fixed Leontief technique, Mandler's (1999a, section 6) claim on generic determinacy of steady-state equilibria fails to verify, and instead shown that generic indeterminacy arises in steady-state equilibria under the same economic model. Here, indeterminacy of a steady-state equilibrium is defined by the existence of a continuum of its nearby steady-state equilibria. This conclusion is strong and remarkable in comparison with the main literature on overlapping-generation indeterminacy, such as Calvo (1978) and Kehoe and Levine (1990), since the main literature typically finds determinate steady states[9] and also a continuum of equilibria near the steady state but those nearby equilibria are not steady states. Moreover, a typical source of overlapping-generation indeterminacy in the main literature is the arbitrariness of initial commodity prices, like Calvo (1978), or the existence of fiat money or a nonzero stock of nominal debt in the initial period, as discussed by Kehoe and Levine (1985, 1990). In contrast, in this paper, a continuum of steady-state equilibria is observed due to the continuum of functional income distributions.

This conclusion has been obtained by the following two features of our model. First, providing a reasonable model of individual optimization program, we have explicitly derived Walras' law from the individual optimization behavior that can make one of the equilibrium equations redundant, and eventually leads us to the opposite conclusion from Mandler's (1999a; section 6) claim on generic determinacy. Remember that Mandler's (1999a; section 6) claim must rely on the existence of non-productive investments, but he has never shown under what kinds of economic mechanisms such investments can support his claim. In contrast, our derived Walras' law can work well even in the cases that a portion of the young's savings is devoted to non-productive investments.

The second feature of our model is introducing capital as a bundle of heterogenous reproducible commodities. This point can be explained by comparing our conclusion with Calvo (1978). As mentioned above, Calvo (1978) finds only determinate steady states under a similar model to ours. Therefore, it would be interesting to find a source of the contrasting results between ours and Calvo's (1978) regarding the features of steady-state equilibria. Note that Calvo (1978) defines capital as a homogenous reproducible good in a two-sector production model, which makes the system of equilibrium equations completely 'decomposed' into two sub-systems. Then, one of the sub-systems yields the stationary level of capital stock and the corresponding stationary production activities, entirely independent of the price system. With the solution of these variables, the remaining sub-system can be solved for the remaining unknowns (the prices), as the numbers of the equations and of the unknowns in the sub-system are

[9] Note that Nishimura and Shimomura (2002, 2006) show the existence of a continuum of steady-state equilibria in dynamic Heckshar-Ohlin international economies. However, the generation of this continuum is due to the infinitely many allocations across two countries of a uniquely determined aggregate capital stock associated with a unique steady-state equilibrium price vector, which corresponds to the unique steady-state equilibrium in our terminology.

identical. However, if capital is defined as a vector of reproducible goods like ours, then the system of steady-state equilibrium equations cannot be 'decomposed', and thus the stationary levels of capital goods and the corresponding production activities cannot be solved independently of the price system and the aggregate demand functions. In such a case, as we have shown in this paper, the number of the equations becomes one less than that of the unknowns in the whole system, unlike the case of Calvo (1978). This would be the source of the opposite conclusions between ours and Calvo's (1978).

Given the generic indeterminacy of steady-state equilibria in the simple Leontief production model, a natural next question would be whether this indeterminacy is robust in more general models. There are at least two interesting more general models: a production model with alternative Leontief techniques to represent economies with the possibility of technical changes; and the von Neumann production model of economies with joint production. Note that the discussion developed in section 5 of Mandler (1999a), referring to both of these models, is irrelevant to this robustness question, as it refers only to the sequential equilibria with non-stationary prices, as in section 3 of Mandler (1999a).

For the model with alternative Leontief techniques, it can be verified that the generic feature of one-dimensional indeterminacy of steady-state equilibria is still observed. Indeed, Yoshihara and Kwak (2023) provides a simple example of OLG production economies with a set of alternative Leontief techniques, in which one-dimensional indeterminacy of non-trivial steady-state equilibria generically emerges. Moreover, it would be easy to derive the same conclusion even in a general model of OLG economies with alternative Leontief techniques.

For the von Neumann model, we conjecture that the generic one-dimensional indeterminacy of steady-state equilibria would be still observed in economies with joint production. At the present stage, we leave it for future research.

Finally, as Mandler's (2002) reference to Morishima (1961) indicates, it would also be interesting to investigate and characterize equilibrium paths in infinite-horison intertemporal economies as argued in the turnpike theorems, given that the continuum of non-trivial steady state equilibria exists.

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6 Appendix: The Existence of Non-trivial Steady-State Equilibrium

In this Appendix, we show that, given an economy $\langle (A, L)\,; \omega_l; u\rangle$, there exists an open subset of available non-negative interest rates such that for every interest rate in this subset, an associated non-trivial steady-state equilibrium exists. By such an existence theorem, it is ensured that the generic indeterminacy discussed in Theorems 1 and 2 is not an empty claim.

Note that if speculative investment were allowed to be non-zero and non-negative under a steady-state equilibrium, then the commodity market clearing condition (b) in Definition 2 would be given by the following form:

$$y + \delta \geqq z(p, w, r) + Ay + \delta,$$

which is also the reduced form of condition (1.2) in Definition 1.

Finally, given that the utility function is strongly monotonic, $\delta \geq \mathbf{0}$ would appear under the non-trivial steady-state equilibrium only when the equilibrium interest rate is zero. Howoever, even when the equilibrium interest rate is zero, $\delta = \mathbf{0}$ is still an optimal action. Therefore, without loss of generality, we may focus on the case of no speculative investment when we discuss the indeterminacy of the non-trivial steady-state equilibrium.

With Definition 2, we can obtain the following existence theorem of the non-trivial steady-state equilibrium in this overlapping economy.

Theorem A1: Let $\langle (A, L); \omega_l; u \rangle$ be an economy as specified above. Then, there exists a *non-trivial steady-state equilibrium* $((p, w, r), y(p, w, r))$ under this economy.

Proof. Let us define $\triangle \equiv \left\{(p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1\right\}$ and $\mathring{\triangle} \equiv \{(p, w) \in \triangle \mid (p, w) > \mathbf{0}\}$. For each $(p, w) \in \triangle$, consider the following opmizaition problem:

$$\max_{(z_b, z_a, y)} u(z_b, z_a)$$

subject to

$$\begin{aligned} pz_b + W &\leqq w\omega_l, \\ pAy &= W, \text{ and} \\ pz_a &\leqq \max\{py - wLy, W\}. \end{aligned}$$

Denote the set of solutions to this optimization problem by $\mathcal{O}(p, w)$.

Take $(z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)$. Then,

$$y(p, w) \in \arg\max \left\{ \max_{y \geq \mathbf{0};\ pAy = W} py - pAy - wLy, 0 \right\}$$

holds. It is also shown that the correspondence $\mathcal{O} : \mathring{\triangle} \twoheadrightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ is non-empty, compact and convex-valued, and upper hemicontinuous.

Let us define the excess demand correspondence $\mathcal{D} : \mathring{\triangle} \twoheadrightarrow \mathbb{R}^n$ by

$$\begin{aligned} &\mathcal{D}(p, w) \\ \equiv\ &\{(z(p, w) - (I - A) y(p, w), Ly(p, w) - \omega_l) \mid (z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)\}. \end{aligned}$$

It can be shown that this correspondence is non-empty, compact and convex-valued, and upper hemicontinuous. By the strong monotonicity of u, the following form of Walras' law holds: for any $(p, w) \in \mathring{\triangle}$ and any $d(p, w) \in \mathcal{D}(p, w)$, $(p, w) \cdot d(p, w) = 0$.

Let us take any price sequence $\left\{\left(p^k, w^k\right)\right\} \subset \overset{\circ}{\triangle}$ such that $\left(p^k, w^k\right) \rightarrow (\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$. Take $d\left(p^k, w^k\right) \in \mathcal{D}\left(p^k, w^k\right)$ for each $\left(p^k, w^k\right)$.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$ with $\overline{w} > 0$. Then, there exists a commodity i such that $\overline{p}_i = 0$. Then, for sufficiently large k, p_i^k is sufficiently close to zero. Then, $z_i\left(p^k, w^k\right)$ is sufficiently large by the strong monotonicity of u. In contrast, $y\left(p^k, w^k\right)$ is bounded by the condition $p^k A y\left(p^k, w^k\right) < w^k \omega_l$. Therefore, for sufficiently large k, $z_i\left(p^k, w^k\right) - y_i\left(p^k, w^k\right) + A_i y\left(p^k, w^k\right) > 0$ should hold, where A_i is the i-th row vector of A. Now, let us define $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \equiv \frac{1}{\lambda}\left(p^k, w^k\right) - \frac{1-\lambda}{\lambda}(\overline{p}, \overline{w})$ for some sufficiently small $\lambda \in (0,1)$. Then, $(p', w') \cdot d\left(p^k, w^k\right) > 0$ holds as $p'_i\left[z_i\left(p^k, w^k\right) - y_i\left(p^k, w^k\right) + A_i y\left(p^k, w^k\right)\right] > 0$ is sufficiently greater.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$ with $\overline{w} = 0$. Then, for sufficiently large k, w^k is sufficiently close to zero. Then, $y\left(p^k, w^k\right)$ must be sufficiently close to zero vector as $p^k A y\left(p^k, w^k\right) < w^k \omega_l$. Thus, for sufficiently large k, $Ly\left(p^k, w^k\right) < \omega_l$ should hold. Now, let us define $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \equiv \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right)$ for some sufficiently small $\varepsilon > 0$. Then,

$$
\begin{aligned}
& (p', w') \cdot d\left(p^k, w^k\right) \\
= & \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right) \cdot \left(z\left(p^k, w^k\right) - (I - A)\, y\left(p^k, w^k\right), Ly\left(p^k, w^k\right) - \omega_l\right) \\
= & \frac{\varepsilon}{1-w^k} p^k \cdot \left[z\left(p^k, w^k\right) - (I - A)\, y\left(p^k, w^k\right)\right] - \varepsilon\left(Ly\left(p^k, w^k\right) - \omega_l\right) \\
= & \frac{w^k}{1-w^k} \varepsilon\left(\omega_l - Ly\left(p^k, w^k\right)\right) - \varepsilon\left(Ly\left(p^k, w^k\right) - \omega_l\right) > 0.
\end{aligned}
$$

In summary, we have shown that for any price sequence $\left\{\left(p^k, w^k\right)\right\} \subset \overset{\circ}{\triangle}$ such that $\left(p^k, w^k\right) \rightarrow (\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$, and for any $d\left(p^k, w^k\right) \in \mathcal{D}\left(p^k, w^k\right)$, there exists $(p', w') \in \overset{\circ}{\triangle}$ such that $(p', w') \cdot d\left(p^k, w^k\right) > 0$ for infinitely many k. Then, by Grandmont (1977, Lemma 1), there exists $(p^*, w^*) \in \overset{\circ}{\triangle}$ such that $z(p^*, w^*) - (I - A)\, y(p^*, w^*) = \mathbf{0}$ and $Ly(p^*, w^*) - \omega_l = 0$. Thus, $y(p^*, w^*) = (I - A)^{-1} z(p^*, w^*)$, and so $y(p^*, w^*) > \mathbf{0}$ by the indecomposability of A, unless $z(p^*, w^*) = \mathbf{0}$. Since $p^* > \mathbf{0}$ and $w^* > 0$, $z(p^*, w^*) \geq \mathbf{0}$ follows from the strong monotonicity of u. Thus, $y(p^*, w^*) > \mathbf{0}$. Then, for $r^* \equiv \frac{p^* y(p^*, w^*) - Ly(p^*, w^*)}{p^* A y(p^*, w^*)} - 1$, $r^* \geqq 0$ holds from $y(p^*, w^*) \in \arg\max\left\{\max_{y \geq \mathbf{0};\ p^* A y = W} p^* y - p^* A y - w^* L y, 0\right\}$. Moreover, it should follow from the optimal behavior and $y(p^*, w^*) > \mathbf{0}$ that

$$
p^* = (1 + r^*)\, p^* A + w^* L.
$$

Thus, there exists a non-trivial steady-state equilibrium $\left((p^*, w^*, r^*), y(p^*, w^*, r^*)\right)$ with $y(p^*, w^*, r^*) = y(p^*, w^*)$. ■

Denote the Frobenius eigenvalue of the matrix A by $(1+R)^{-1} \in (0,1)$. Then, by Theorem A1 and Theorem 1, we have the following existence theorem.

Theorem A2: Let $\langle (A,L); \omega_l; u \rangle$ be an economy as specified above. Let $((p^*, w^*, r^*), y(p^*, w^*, r^*))$ be a non-trivial steady-state equilibrium. Then, there exists an open neighborhood $\mathcal{N}(r^*) \subseteq [0,R)$ of r^* such that there exists a non-trivial steady-state equilibrium

$$((p(r), w(r), r), y(p(r), w(r), r))$$

for every $r \in \mathcal{N}(r^*)$.

Proof. Let us define a continuously differentiable function $F : \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times [0,R) \times \mathbb{R}_+^n \to \mathbb{R}^{2n}$ as:

$$F(\overline{p}, w, r, y) = \begin{bmatrix} z(p,w,r) - [I-A]y \\ \overline{p}_{-n} - (1+r)\overline{p}A_{-n} - wL_{-n} \\ Ly - \omega_l \end{bmatrix}.$$

Let (p^*, w^*, r^*, y^*) be a non-trivial steady-state equilibrium, whose existence is ensured by Theorem A1. Moreover, without loss of generality, assume it is regular.[10] Then, the Jacobian of F at (p^*, w^*, r^*, y^*) is given by $\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))$ as follows:

$$\begin{aligned} &\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*)) \\ = &\begin{bmatrix} [A-I] & \boldsymbol{D}_{\overline{p}} z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_w z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_r z(\overline{p}^*, w^*, r^*) \\ \mathbf{0} & I_{n-1} - (1+r)A_{-n}^T & -L_{-n}^T & -(\bar{p}A_{-n})^T \\ L & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

As (p^*, w^*, r^*, y^*) is regular, it follows that $rank\left[\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))\right] = 2n$.

Then, by the implicit function theorem, there exist an open neighborhood $\mathcal{N}(r^*) \subset [0,R)$ of r^* and also an open neighborhood $\mathcal{M}(\overline{p}^*, w^*, y^*) \subset \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+^n$ of $(\overline{p}^*, w^*, y^*)$ such that there exists a continuous single-valued mapping $\eta : \mathcal{N}(r^*) \to \mathcal{M}(\overline{p}^*, w^*, y^*)$ such that for any $r' \in \mathcal{N}(r^*)$, there exists $(\overline{p}', w', y') = \eta(r')$ with $F(\overline{p}', w', r', y') = \mathbf{0}$. By the definition of the mapping F, $F(\overline{p}', w', r', y') = \mathbf{0}$ implies that $\overline{p}' \cdot (z(\overline{p}', w', r') - [I-A]y') + w' \cdot (Ly' - \omega_l) = 0$. As $\overline{p}'_{-n} = (1+r')\overline{p}'A_{-n} + w'L_{-n}$, it also follows that $1 = (1+r')\overline{p}'A_n + w'L_n$. Thus, $\overline{p}' = (1+r')\overline{p}'A + w'L$ holds, which implies that $(\overline{p}', w', r', y')$ is a non-trivial steady-state equilibrium associated with $r' \in \mathcal{N}(r^*)$. ■

[10] As shown in the proof of Theorem 2, the regularity of the equilibrium (p^*, w^*, r^*, y^*) is indeed verified.