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# Revealed Preference Tests for Price Competition in Multi-product Differentiated Markets

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**Abstract**

Assumptions of competitive structure are often crucial for marginal cost estimation and counterfactual predictions. This paper introduces tests for price competition among multi-product firms. The tests are based on the firm's revealed preference (revealed profit function). In contrast to other approaches based on estimated demand functions such as conduct parameter estimation, the proposed tests do not require any instrumental variables, even though the models can accommodate structural error terms. In this paper, I employ a demand structure introduced by Nocke and Schutz (2018), the discrete/continuous choice model, which nests the multinomial logit demand and CES demand functions. Any price and quantity data can be rationalized by price competition under a discrete/continuous choice model and increasing marginal costs. Adding more assumptions to the demand function, such as logit, CES, or the *co-evolving* and *log-concave* property produces some falsifiable restrictions.

**Keywords:** revealed preference, multi-product, conduct, discrete/continuous

## 1 Introduction

In the industrial organization literature, we often assume specific competitive structures such as price competition or quantity competition. In many cases, competitive structure assumptions are crucial for empirical research. For instance, we often back out marginal costs from first-order conditions based on estimated demand functions and competitive structures. Results of counterfactual analysis, which often provide the main policy implication in research

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with structural models, also depend on the imposed competitive structures. Even though we can obtain parameter estimates in a structural model that fits the data best, the structural model itself could possibly not fit the data. That is, the data might not be rationalized by the model for any possible parameters. Furthermore, the data might not be rationalized by any realizations of structural error terms. This is because some data points might be outside the support of the structural model. In this paper, I provide a systematic method to detect such inconsistency between the data and price competition among single- or multi-product firms under a certain class of demand functions.

Regarding consumer behavior, Afriat (1967) shows that finite data satisfy GARP if and only if they are rationalized by utility maximization given a finite set of price vectors. That is, if the data violate GARP, then they cannot be explained by any (locally non-satiated) utility functions. Brown and Matzkin (1996) extend this idea to the general equilibrium framework. Carvajal et al. (2013) apply the idea to Cournot competition, and show that Cournot rationalizability can be checked by the existence of parameters that satisfy some inequalities. Carvajal et al. (2014) introduce a few variants of Carvajal et al. (2013): a test for multi-market-contact Cournot competition and a test for price competition in a differentiated market. However, they only focus on price competition where each firm produces a single product. Price competition with multi-product firms is often examined in the empirical industrial organization literature (e.g., Berry et al. (1995), Goldberg (1995)). One of the main difficulties in extending Carvajal et al.'s (2014) test to competition among multi-product firms arises from the substitution effects among the same firm's products (or cannibalization effects). We can circumvent such a difficulty by employing an important class of demand structure, namely the discrete/continuous choice model introduced by Nocke and Schutz (2018), which nests the multinomial logit demand function and constant elasticity of substitution (CES) demand function as special cases.

In order to test the competitive structure, we can also estimate the *conduct parameter*. Bresnahan (1982) shows that we can identify the conduct parameter in an industry from a rotation of the demand function over time (see Bresnahan (1989) for its estimation and applications). Researchers have recently estimated the extent to which firms internalize other firms' profits, a measure that is closely related to the conduct parameter (e.g., Miller and Weinberg (2017) and Sullivan (2016)). Alternatively, if we have data on the cost structure, we could compare marginal costs backed out from the model with the actual cost data since different competitive structures yield different first order conditions and, in turn, different estimates of marginal costs (e.g., Wolfram (1999)). The revealed preference test examined in this paper provides an alternative approach with advantages and disadvantages. The first advantage is that the revealed preference test in this paper does not require any IVs while both conduct parameter estimation and the Wolfram (1999) approach require appropriate IVs. The reason we do not need IVs is that we do not estimate parameters to test the model, but rather check restrictions that are valid for any parameter values. This is analogous to Afriat (1967)'s theorem, which characterizes a set of data restrictions satisfied if consumers maximize their own utility, regardless of the underlying utility function specification. Second, we only need the market-level price and quantity data, and no other product characteristics, to implement the test. Due to this parsimonious data requirement, this test could be used as a pretest/sanity check before detailed estimation.

A disadvantage of the test is that it is a joint test of competitive structure and demand/cost functions. Therefore, rejection of the model might imply other types of competition under a discrete/continuous demand structure, price competition under other demand functions, or other types of competition under other demand functions. Second, even though the discrete/continuous choice model is general enough to include the logit and CES demand functions as special cases, it still has the independence-to-irrelevant-alternative (IIA) property. Therefore, the main theorem in this paper does not hold for the random coefficient logit model (e.g., Berry et al. (1995)). Another issue is that cost functions are assumed to be invariant over time in the tested model, even though the constant marginal costs are not implied since cost functions are assumed to be convex but time-invariant. Therefore, the test should be implemented for short-panel data, for which the cost structure is not supposed to change during the time range. In practice, if a researcher has a long panel, then the data can be divided into many short panels, the test can be implemented for each short data segment, and the rejection ratio can be reported (e.g., Carvajal et al. (2014)). Incorporating the observed cost shifter alleviates this issue, as shown in Section 3.

In terms of the power of tests, any data would satisfy the rationalizability condition of the price competition under the general discrete/continuous demand function. This result is inconsistent with the findings of Carvajal et al. (2013) and Carvajal et al. (2014). The key difference is that they consider demand changes due to a common shock for different firms.[1] Naturally, we can obtain falsifiable restrictions by imposing a similar additional restriction that is compatible with the discrete/continuous demand function. We can also obtain falsifiable restrictions by restricting the underlying demand function to a sub-class of discrete/continuous demand structures, which still nests both logit and CES as its special cases. This also implies that price competition under a logit or CES demand function is falsifiable.

[1]Carvajal et al. (2013) consider Cournot competition in the homogenous goods market, where demand shock is common to all firms. Carvajal et al. (2014) consider multi-market contact Cournot competition and differentiated price competition. With differentiated price competition, they introduce additional restrictions that capture the idea of a common demand shock.

In general, we can check the set of restrictions by evaluating a loss function similar to those for moment inequality estimations. In principle, therefore, the revealed preference tests in this article share some computational issues with moment inequality estimations. However, we can characterize the set of restrictions as a set of linear constraints over parameters by focusing on the logit demand function and considering a slightly modified data requirement, which must always be satisfied when researchers estimate logit demand functions. Then, we can implement the test through standard algorithms for linear constraints.

The remainder of the paper is organized as follows. I introduce the main model and its special cases in Section 2. I first exemplify a revealed preference test under a logit demand function and then formalize and generalize the result. In Section 3, I discuss some extensions of the test with (i) additional demand restrictions discussed in earlier research, (ii) observed cost shifters, and (iii) the possibility of collusive conduct among firms. I discuss the algorithms for the tests in Section 4, and provide a summary in Section 5.

# 2 The model

In this paper, I consider a standard competition framework for a differentiated market, where each firm produces different products. The products can be similar but not completely the same. The demand functions are assumed to change over time, potentially because of a change in consumer tastes or product characteristics. The characteristics might or might not be observed by the econometrician. I denote $\mathcal{J} = \{1, 2, ..., J\}$ as a set of products and $Q_{j,t} : R_+^J \to R_+$ as a demand function of product $j \in \mathcal{J}$ at time $t \in \{1, ..., T\} \equiv \mathcal{T}$. The demand function is assumed to be in a class of *discrete/continuous* models, which is explained later. Firm $f \in \{1, ..., F\}$ produces a set of products $\mathcal{J}_f \subset \mathcal{J}$ s.t. $\mathcal{J}_f \cap \mathcal{J}_g = \phi$ for $f \neq g$ and denote $J_f = |\mathcal{J}_f|$. The cost function of product $j \in \mathcal{J}$, $C_j : R_+ \to R$ , is assumed to be convex and twice continuously differentiable.[2] In this section, I focus on time-invariant cost functions, which serve a similar purpose as time-invariant preference in Afriat (1967).[3] Then, the profit function for firm $f$ at time $t$ is written as $\pi_{f,t}(p) = \sum_{j \in \mathcal{J}_f} \{Q_{j,t}(p)\, p_j - C_j(Q_{j,t}(p))\}$.

Let the observed price and quantity be as follows: $\{\bar{p}, \bar{q}\}$ where $\bar{x} = (\bar{x}_1', ..., \bar{x}_T')'$ and $\bar{x}_t = (\bar{x}_{1,t}, ..., \bar{x}_{J,t})'$ for $x = p, q$.[4] In the following, I introduce tests of whether a set of data

[2]Even though the cost functions here are more general than linear cost functions, which are often assumed in empirical work, the additive separable cost functions are still restrictive in the context of multiple products because it excludes the economy of scope in the production process. In Section 3, I incorporate non-separable and convex cost functions. In the main part, I focus on additive separable cost functions for each product, to permit simpler interpretations.

[3]In Section 3, I discuss an extension with time-variant and observed cost shifters.

[4]Throughout the paper, I distinguish observed data on prices and quantities from arbitrary values of price

$\{\bar{p}, \bar{q}\}$ can be rationalized by price competition.

**Definition 1.** A set of data $\{\bar{p}, \bar{q}\}$ is *rationalizable* by price competition if there exist demand and cost functions under which $\{\bar{p}_t, \bar{q}_t\}$ is generated as a result of Nash equilibrium of price competition for any $t \in \mathcal{T}$. When $\{\bar{p}, \bar{q}\}$ can be rationalized by price competition, $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable.

I primarily utilize first-order conditions of profit functions and cost convexity to derive testable data restrictions that should be satisfied regardless of the parameter values and structural error terms. Using the profit function defined above, the first-order condition w.r.t. $p_j$ is written as

$$0 = Q_{j,t}(p) + \sum_{k \in J_f} \{p_k - C_k'(Q_{k,t}(p))\} \frac{\partial Q_{k,t}(p)}{\partial p_j}. \tag{1}$$

## 2.1 Logit Demand Function

Before proceeding to the main result with a general specification, I demonstrate that some data cannot be explained by price competition with the logit demand function, which is a special case of the discrete/continuous model. By using a logit demand function,

$$Q_{j,t}(p) = M_t \frac{\exp(v_{jt} - \alpha p_{jt})}{1 + \sum_k \exp(v_{kt} - \alpha p_{kt})}$$

for some $M_t$, $\alpha \in R_+$ and $(v_{j,t})_{j \in \mathcal{J}} \in R^J$, the first-order condition is rewritten as follows:[5]

$$0 = Q_{j,t}(p) - \left\{p_j - C_j'(Q_{j,t}(p))\right\} \alpha Q_{j,t}(p) + \sum_{k \in J_f} \{p_k - C_k'(Q_{k,t}(p))\} \frac{\alpha}{M_t} Q_{k,t}(p) Q_{j,t}(p)$$

By rearranging it, we obtain the following equation:

$$p_j - C_j'(Q_{j,t}(p)) = \frac{1}{\alpha} + \frac{1}{M_t} \sum_{k \in J_f} \{p_k - C_k'(Q_{k,t}(p))\} Q_{k,t}(p). \tag{2}$$

**Example 1. Between-Firm Restriction.**

The first example shows that the logit specification generates between-firm restrictions on data to be rationalized. To emphasize this, I focus on single-product firms 1 and 2, each

and quantity by denoting the data as $\bar{p}$ and $\bar{q}$ and the arbitrary values as $p$ and $q$. This helps clarify proofs in the Appendix.

[5] The following argument holds more generally with the time-variant $\alpha_t$, instead of the time-invariant $\alpha$. I use the time-invariant $\alpha$ simply because it is used more commonly in the literature.

of which produces product 1 and 2, respectively. Then, the first-order condition becomes

$$p_j - C_j' \left(Q_{j,t}\left(p\right)\right) \quad = \quad \frac{1}{\alpha} + \frac{1}{M_t}\left\{p_j - C_j'\left(Q_{j,t}\left(p\right)\right)\right\} Q_{j,t}\left(p\right). \tag{3}$$

for $j = 1,\, 2$. By cancelling $1/\alpha$, we obtain the following necessary condition for equilibrium:

$$\frac{p_1 - C_1'\left(Q_{1,t}\left(p\right)\right)}{p_2 - C_2'\left(Q_{2,t}\left(p\right)\right)} = \frac{M_t - Q_{2,t}\left(p\right)}{M_t - Q_{1,t}\left(p\right)}. \tag{4}$$

Suppose that we observe the following data: $\left(\bar{p}_{j,\tau},\, \bar{q}_{j,\tau}\right)_{j=1,2,\,\tau=s,t}$ s.t. $\bar{p}_{j,\,t} = \bar{p}_{j,\,s} = \bar{p}_j$ for $j = 1,\, 2$ and $\bar{q}_{2,\,t} > \bar{q}_{1,\,t} = \bar{q}_{1,\,s} > \bar{q}_{2,\,s}$. The RHS for time $t$ and $s$ gives

$$\frac{M_t - \bar{q}_{2,\,t}}{M_t - \bar{q}_{1,\,s}} < 1 < \frac{M_s - \bar{q}_{2,\,s}}{M_s - \bar{q}_{1,\,s}}. \tag{5}$$

However, the LHS for time $t$ and $s$ gives

$$\frac{\bar{p}_1 - C_1'\left(\bar{q}_{1,\,t}\right)}{\bar{p}_2 - C_2'\left(\bar{q}_{2,\,t}\right)} \geq \frac{\bar{p}_1 - C_1'\left(\bar{q}_{1,\,s}\right)}{\bar{p}_2 - C_2'\left(\bar{q}_{2,\,s}\right)}. \tag{6}$$

Eqs. (5) and (6) show that the data do not satisfy (4). Thus the data cannot be explained by price competition under a logit demand function. The explanation is as follows. In the logit specification, a change in price and quantity over time can be explaiend by change in values of the products, $v_{j,t}$, or market size, $M_t$. If $v_{j,t}$ changes, the price $p_{j,\,t}$ changes in the same direction as $v_{j,\,t}$. If prices do not change over time, as in the example, a change in quantity can be explained by the market size. However, change in the market size is common to both products. Therefore, the quantities of both products should move in the same direction.

**Example 2. Within-Firm Restriction.**

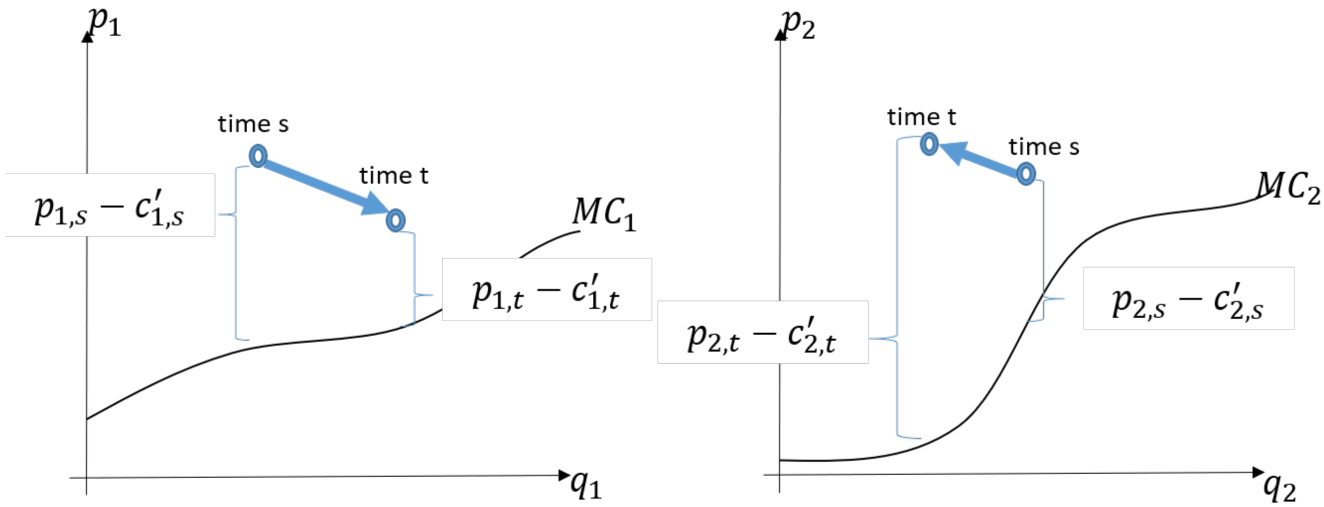
The second example shows that the logit demand function generates within-firm restrictions. In eq. (2), the RHS applies in common to the goods produced by the same firm. Therefore,

$$p_j - C_j'\left(Q_{j,t}\left(p\right)\right) = p_k - C_k'\left(Q_{k,t}\left(p\right)\right) \text{ for any } j,\, k \in \mathcal{J}_f \tag{7}$$

must hold at equilibrium. In this paper, I call this property the "common mark-up property." [6] The model is rejected by the common mark-up property together with the increasing marginal cost assumption, given the following data: $\left(\bar{p}_{j,\tau},\, \bar{q}_{j,\tau}\right)_{j=1,2,\,\tau=s,t}$ s.t. $\{1,2\} \subset \mathcal{J}_f$, $\bar{p}_{1,s} > \bar{p}_{1,t}$, $\bar{p}_{2,s} < \bar{p}_{2,t}$, $\bar{q}_{1,s} < \bar{q}_{1,t}$, and $\bar{q}_{2,s} > \bar{q}_{2,t}$. That is, the price and quantity of good

[6] More generally, Nocke and Schutz (2018) call it the "common $\iota-$markup property" under the discrete/continuous choice model.

Figure 1: Example: Logit Demand Function




1 and those of good 2 move in the opposite direction (see Fig. 1). Suppose that the data satisfy eq.(7) at time $s$. If the marginal costs are (weakly) increasing with own quantity, then $\bar{p}_{1,t} - C_1'(\bar{q}_{1,t}) < \bar{p}_{1,s} - C_1'(\bar{q}_{1,s}) = \bar{p}_{2,s} - C_2'(\bar{q}_{2,s}) < \bar{p}_{2,t} - C_2'(\bar{q}_{2,t})$. Therefore, eq.(7) cannot be satisfied at time $t$. Thus, these data, $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\,\tau=s,t}$, cannot be explained by (a repetition of static) price competition under logit demand functions. This means these data cannot be explained by any set of parameters, $\alpha$, $m_t$, and $v_{j,t}$ and non-parametric cost functions, $C_j(\cdot)$.

**Properties of revealed preference tests** The above examples highlight some important features of revealed preference tests for competitive structures.

**IVs not needed** With the logit specification, one can better understand an underlying mechanism of the revealed preference test by comparing it with an alternative procedure to check the competitive structure. A parameter of the demand function, $\alpha$, is often estimated from aggregated data (with the use of IVs to address unobserved heterogeneity potentially correlated with prices), and $\delta$'s are backed out from the first-order condition. Subsequently, we could check whether the obtained $\delta$'s are reasonable. Alternatively, in the revealed preference test, a similar procedure is used for any possible $\alpha > 0$, instead of an estimated $\alpha$ determined by IVs. Therefore, we do not need IVs with unobserved heterogeneity that is potentially correlated with price or some other variables.

**Interpretation of rejection** Note that rejection or acceptance is not probabilistic even if the model has (only) the structural error term in the logit demand function. When we estimate logit demand functions from aggregate data, $v_{j,t}$ is decomposed as $v_{j,t} = x'_{j,t}\beta + \xi_{j,t}$ where $x_{j,t}$ is a vector of product $j$'s observed characteristics, $\xi_{j,t}$ represents the unobserved characteristics, and $\beta$ is a vector of parameters. In the logit demand estimation, $\xi_{j,t}$ is treated as a structural error term. However, eq.(7) should be satisfied regardless of what values of $\xi_{j,t}$ are realized, as long as firms compete on price under logit demand functions (recall that I did not impose any assumptions on $v_{j,t}$). This is because each firm (but not the econometrician) is assumed to know what $(\xi_{j,t})_{j\in\mathcal{J}}$ is realized, as is often assumed in the empirical IO literature. Therefore, any rejection of the model cannot be attributed to a peculiar realization of structural error terms.

**(No) data restrictions in each assumption** In this article, revealed preference tests are joint tests of demand and cost specifications and the competitive structure. However, it is worth noting that each of them by itself cannot be rejected by any data, $\{\bar{p}, \bar{q}\}$, but can only be rejected together. Assuming only a logit demand function, for any data $(\bar{p}_{j,t}, \bar{q}_{j,t})_{j\in\mathcal{J}}$ at each $t$, we can back out the corresponding $(v_{j,t})_{j\in\mathcal{J}}$ by an inversion of the market share function as in Berry (1994). Thus, logit demand can fit the data since any changes in data over time can be captured by changes in $(v_{j,t})_{j\in\mathcal{J}}$ over time. Regarding the assumption of price competition and cost functions, any data can be rationalized by price competition under a more general demand function and convex time-invariant cost function, as explained in Section 2.2. This emphasizes that each assumption in this article is not trivially restrictive, especially when we have only price and quantity data.

In the following part, I provide a set of inequalities as a systematic method to detect data inconsistent with price competition, and show that such conditions are sufficient for rationalization by price competition. Instead of the logit demand function, I employ a class of demand functions by Nocke and Schutz (2018) that nests the logit demand function and CES demand function.

## 2.2 Discrete/Continuous Demand Function

In the following, I employ the discrete/continuous demand function introduced by Nocke and Schutz (2018), where the demand function for product $j$ is written as

$$Q_j(p) = m\frac{-h'_j(p_j)}{h_0 + \sum_{k\in I} h_k(p_k)},$$

where $h_j(\cdot)$ is decreasing and log-convex for every $j$, and $m$ is a positive constant. An important example of this demand function is the logit model $h_j(p_j) = \exp(v_j - \alpha p_j)$ and

$m = M/\alpha$, where $v_j \in R$ is the value of good $j$, $\alpha > 0$ is the coefficient for prices, $M > 0$ is the size of the market, and $h_0$ is the exponentiated value of the outside option.[7] Another important example is the CES model $h_j(p_j) = a_j p_j^{1-\sigma}$ and $m = I/(\sigma - 1)$, where $I$ is the income level of the consumer and $\sigma$ is the elasticity of substitution ($\sigma > 1$).

In this paper, I utilize the fact that we can express the partial derivatives of the discrete/continuous demand function in a simple form:

$$\begin{aligned}\frac{\partial Q_{k,t}(p)}{\partial p_j} &= m \frac{-h'_{k,t}(p_k)}{\left(h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)\right)^2} \left(-h'_{j,t}(p_j)\right). \\ &= m^{-1} Q_{k,t}(p) \cdot Q_{j,t}(p) \quad \forall k \neq j\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Q_{j,t}(p)}{\partial p_j} &= m_t \frac{-h''_{j,t}(p_j)}{h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)} + m_t \left(\frac{-h'_{j,t}(p_k)}{h_{0,t} + \sum_{k \in I} h_{k,t}(p_k)}\right)^2 \\ &= -Q_{j,t}(p) \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} + m_t^{-1} \left(Q_{j,t}(p)\right)^2 \\ &= -m_t^{-1} Q_{j,t}(p) \left\{ m_t \frac{h''_{j,t}(p_k)}{-h'_{j,t}(p_k)} - Q_{j,t}(p) \right\}.\end{aligned}$$

It is worth noting that $-m_t h''_{j,t}(p_k) / h'_{j,t}(p_k) - Q_{j,t}(p)$ is positive because of the log-convexity of $h_j(\cdot)$.[8]

[7] As discussed by Nocke and Schutz (2018), a discrete/continuous choice model with an outside option can be normalized to a discrete/continuous choice model without an outside option, $\tilde{Q}_j(p) = -m\tilde{h}'_j(p_j) / \left(\sum_{k \in I} \tilde{h}_k(p_k)\right)$, by letting $\tilde{h}_j(p_j) = \frac{1}{J} h_0 + h_j(p_j)$. In this paper, I express $h_0$ explicitly in order to intuitively explain the results in the later parts.

[8] The log-convexity implies

$$\frac{h''_j(p)}{-h'_j(p)} > \frac{-h'_j(p)}{h_j(p)} \left(> \frac{-h'_j(p)}{h_0 + \sum h_k(p)}\right).$$

With the above expression, the FOC w.r.t. $p_j$ is written as follows:

$$\begin{aligned} 0 &= 1+\sum_{k\in J_f}\left\{p_k-C_k'\left(Q_{k,t}\left(p\right)\right)\right\}\frac{\partial Q_{k,t}\left(p\right)}{\partial p_j}\frac{1}{Q_{j,t}\left(p\right)} \\ &= 1-m_t^{-1}\left\{p_j-C_j'\left(Q_{j,t}\left(p\right)\right)\right\}\left\{m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)}-Q_{j,t}\left(p\right)\right\} \\ &\quad +m_t^{-1}\sum_{k\in J_f,\,k\neq j}\left\{p_k-C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right) \\ &= m_t-\left\{p_j-C_j'\left(Q_{j,t}\left(p\right)\right)\right\}m_t\frac{h_{j,t}''\left(p_k\right)}{-h_{j,t}'\left(p_k\right)}+\sum_{k\in J_f}\left\{p_k-C_k'\left(Q_{k,t}\left(p\right)\right)\right\}Q_{k,t}\left(p\right) \end{aligned}$$

Therefore, if the data $\{\bar{p},\, \bar{q}\}$ are generated by price competition with an (unknown) discrete/continuous demand function, there exists $\alpha_{jt}$, $\delta_{jt}$, which corresponds to $-h_{j,t}''\left(\bar{p}_j\right)/h_{j,t}'\left(\bar{p}_j\right)$ and $C_j'\left(\bar{q}_{j,t}\right)$, respectively, such that

$$0 = m_t-\left\{\bar{p}_j-\delta_{j,t}\right\}m_t\alpha_{j,t}+\sum_{k\in J_f}\left\{\bar{p}_k-\delta_{j,t}\right\}\bar{q}_{k,t}. \tag{8}$$

On the other hand, since $\delta_{j,t}$ corresponds to $C_j'\left(\bar{q}_{j,t}\right)$ and $C_j'\left(\cdot\right)$ is assumed to be increasing, $\delta_{j,t}$ must be greater than $\delta_{j,s}$ $(s\neq t)$ if $\bar{q}_{j,t}$ is greater than $\bar{q}_{j,s}$. This is summarized as an inequality:

$$0\leq\left(\delta_{j,s}-\delta_{j,t}\right)\left(\bar{q}_{j,s}-\bar{q}_{j,t}\right). \tag{9}$$

Combining eq.(8) and eq.(9), we obtain a set of necessary conditions for the data to be rationalized by the model. Furthermore, the conditions are also sufficient for rationalization. They are summarized in the following theorem.

**Theorem 1.** *(Discrete/Continuous): The set of observations $\{\bar{p},\, \bar{q}\}$ is Bertrand-rationalizable under a convex cost function and a discrete/continuous demand function if and only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and $m_t$ for any $t\in\mathcal{T}$ and $j\in\mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t}>0$, $\delta_{j,t}>0$, $m_t>0$;
2. $0=m_t-\left\{\bar{p}_{j,t}-\delta_{j,t}\right\}m_t\alpha_{j,t}+\sum_{k\in J_f}\left\{\bar{p}_{k,t}-\delta_{k,t}\right\}\bar{q}_{k,t}$; *and*
3. $0\leq\left(\delta_{j,t'}-\delta_{j,t}\right)\left(\bar{q}_{j,t'}-\bar{q}_{j,t}\right)$.

The first set of conditions is derived from the underlying specifications of the demand and cost functions: $\alpha_{j,t}>0$ from the assumption that $h_j$ is decreasing and log-convex, $\delta_{j,t}>0$ from the increasing cost functions, and $m_t>0$ from the assumption that the quantity of each good is non-negative. The proof of sufficiency consists of two steps. First, given $\{\alpha_{j,t}\}$ and $\{\delta_{j.t}\}$, which satisfy the conditions, I construct demand functions $\{\bar{Q}_{j,t}(\cdot)\}$ and cost functions $\{\bar{C}_j(\cdot)\}$ to satisfy $-\bar{h}''_{j,t}(\bar{p}_j)/\bar{h}'_{j,t}(\bar{p}_j) = \alpha_{j,t}$ and $\bar{C}'_j(\bar{q}_{j,t}) = \delta_{j.t}$.[9] Then, the data $\{\bar{p}, \bar{q}\}$ satisfy the first-order conditions under the reconstructed demand and cost functions. In the second step, I show that the first-order conditions are a sufficient condition for profit maximization given the other firms' prices and the reconstructed demand cost functions. This result is not trivial since the profit function does not satisfy quasi-concavity. In this paper, sufficiency is proved by the unique solution of the first-order conditions. The uniqueness is derived from the unique "common $\iota$-markup" and a mapping from $\iota$-markup to price vectors as in Nocke and Schutz (2018). See the appendix for the full proof.[10]

## 2.3 Special Cases: Logit and CES

For more restrictive specifications, such as logit or CES demand functions, we can easily derive the necessary condition for data to be rationalized by the models, by simply adding restrictions to the second condition in the above tests. Sufficiency of the restriction, however, is less trivial. In the proof of sufficiency in Theorem 1, I reconstruct the demand function as $\{\bar{Q}_{j,t}(\cdot)\}$, which still nests the logit demand function, but not CES. Such a reconstruction is sufficient for Theorem 1 since the reconstructed demand function $\{\bar{Q}_{j,t}(\cdot)\}$ is in the class of demand functions we are interested in. To test a model with the logit demand, we can apply a similar reconstruction of $\{\bar{Q}_{j,t}(\cdot)\}$ and the remaining is proved analogously. In contrast, such a reconstruction is no longer valid for a model with CES demand. Therefore, a demand function is reconstructed in a different way, and the sufficiency of the first-order condition is proved by a slightly different method due to the different construction of the demand function.

The modified tests mentioned above are articulated in the following propositions (the specifications and results are summarized in Table 1 in the appendix).

**Proposition 1.** *(Logit) The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and logit demand functions* ***if and only if*** *there exist real numbers $\alpha_t$, $\delta_{j,t}$, and $m_t$ for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following hold:*

[9] In the proof of sufficiency, the constructed demand functions $\{\bar{Q}_{j,t}(\cdot)\}$ can be different from the actual demand functions $\{Q_{j,t}(\cdot)\}$. For instance, in the proof of Theorem 1, even if the data $\{\bar{p}, \bar{q}\}$ are actually generated from CES demand, which is a special case of the discrete/continuous choice model, the reconstructed $\{\bar{Q}_{j,t}(\cdot)\}$ can be a non-CES demand as long as it is another special case of the discrete/continuous choice model. This point is further discussed in the next subsection.

[10] Notably, the second condition is generally not linear because of the interaction of $\delta_{j,t}$ and $\alpha_{j,t}$, which contrasts with the finding of Carvajal et al. (2013, 2014). Thus, we cannot use algorithms for linear programming. One way to implement the above test is to consider an algorithm similar to moment inequalities.

1. $\alpha > 0$, $\delta_{j,t} > 0$, $m_t > 0$ ;

2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*

3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

In the above statement, $\alpha_t$ is allowed to vary over time for the sake of generality. We can easily replace $\alpha_t$ with time-invariant $\alpha$. Such a simplified version is proved analogously to Proposition 1.

**Proposition 2.** *(CES): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and CES demand functions if and only if there exist real numbers $\sigma_t$, $\delta_{j,t}$, and $m_t$ for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$, such that the following hold:*

1. $\sigma_t > 1$, $\delta_{j,t} > 0$, $m_t > 0$ ;

2. $0 = m_t - \frac{\bar{p}_{j,t} - \delta_{j,t}}{\bar{p}_{j,t}} m_t \sigma_t + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*

3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

As explained earlier, this necessary condition is derived as a special case of (the necessity part of) Theorem 1. However, the sufficiency is not derived from Theorem 1 since the reconstructed demand function should be also CES instead of arbitrary discrete/continuous demand. See the Appendix for the proof.

## 2.4 Falsifiability

Theorem 1 characterizes the necessary and sufficient condition for the data to be rationalized by the model of price competition under discrete/continuous demand functions and time-invariant convex cost functions. Meanwhile, readers might wonder how restrictive the conditions are. It turns out that the restriction in Theorem 1 is so loose that any data can be rationalized by the model with the general discrete/continuous demand functions. This may be surprising, considering that even the general discrete/continuous choice model satisfies the IIA property. Any changes in price and quantity along a fixed discrete/continuous demand function must satisfy the IIA property, while demand functions themselves are allowed to change over time with the model in Theorem 1. To clearly understand how any data satisfy the restrictions, consider the following. For any given $\delta_{j,t}$ and $m_t$, the remaining parameter $\alpha_{j,t}$, which characterizes the demand functions, can be determined only through the first-order condition w.r.t. $p_{j,t}$, independently of the first-order condition w.r.t. $p_{k,s}$, where $k \neq j$ or $s \neq t$. [11] Thus, for any data, we can find corresponding $\alpha_{j,t}$, $\delta_{j,t}$, $m_t$, and then, the sufficiency implies that any data are rationalized by the model.

**Corollary 1.** *Any data, $\{\bar{p}, \bar{q}\}$, are Bertrand-rationalizable under convex cost functions and discrete/continuous demand functions.*

Even though price competition under the general discrete/continuous choice model is not falsifiable, a model can be falsifiable under a more restrictive demand model such as the logit demand function as shown in the example. This naturally raises a question: How general is this falsifiability? In the following, I show that a subclass of discrete/continuous demand functions that nests both the logit and CES demand is falsifiable. Consider a discrete/continuous demand function generated by $h_{j,t}(\cdot)$ such that

$$\frac{h''_{j,t}(p_j)}{-h'_{j,t}(p_j)} = \frac{1}{a_t p_j + b_t}$$

for some $1 > a_t \geq 0$ and $b_t \geq 0$. Now, we can express the logit and CES demand function by setting $a_t = 0$ and setting $b_t = 0$, respectively. I call such a demand function a discrete/continuous demand function with HARA $h$ since $h$ is characterized as analogous to a hyperbolic absolute risk averse vNM utility function, which nests CARA and CRRA as special cases. First, I introduce a modified version of the necessary and sufficient condition for data to be rationalized by price competition under the modified specification.

**Proposition 3.** *The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under convex cost functions and discrete/continuous demand functions with HARA $h$ if and only if there exist real numbers $a_t$, $b_t$, $\delta_{j,t}$, and $m_t$ for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $1 > a_t \geq 0$, $b_t \geq 0$, $\delta_{j,t} > 0$, $m_t > 0$ *;*
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \frac{1}{a_t p_{j,t} + b_t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$*; and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$*.*

Note that the modified model is falsifiable, i.e., the model can be rejected given certain data. By the second condition in the set of restrictions, the data must satisfy

$$\frac{\bar{p}_{j,t} - \delta_{j,t}}{a_t \bar{p}_{j,t} + b_t} = \frac{\bar{p}_{k,t} - \delta_{k,t}}{a_t \bar{p}_{k,t} + b_t}$$

for all $j$, $k \in \mathcal{J}_f$ and all $t \in \mathcal{T}$. The following example shows how this applies.

[11] In contrast, under the logit demand, $\alpha_t$ is common to all goods so that data restrictions on different products are linked to each other.

**Example 3. Discrete/Continuous with HARA** $h$

Consider a case in which one firm produces products 1, 2, and 3 and generates the data $\{\bar{p},\, \bar{q}\}$ such that $\bar{p}_{j,t} = \bar{p}_{j,s} \equiv \bar{p}_j$ and $\bar{q}_{j,t} = \bar{q}_{j,s} \equiv \bar{q}_j$ for $j = 1,\, 2$ and for some $t,\, s \in \mathcal{T}$, and $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

Then, the above equality is rewritten as follows:

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_t/a_t} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_t/a_t} = \frac{\bar{p}_{3,t} - \delta_{3,t}}{\bar{p}_{3,t} + b_t/a_t} \tag{10}$$

and

$$\frac{\bar{p}_1 - \delta_1}{\bar{p}_1 + b_s/a_s} = \frac{\bar{p}_2 - \delta_2}{\bar{p}_2 + b_s/a_s} = \frac{\bar{p}_{3,s} - \delta_{3,s}}{\bar{p}_{3,s} + b_s/a_s}. \tag{11}$$

Note that the equalities for goods 1 and 2 in eqs. (10) and (11) imply that $b_t/a_t = b_s/a_s \equiv b/a$. Therefore, all the terms in eqs. (10) and (11) must be the same. Thus, for good 3,

$$\frac{\bar{p}_{3,t} - \delta_{3,t}}{\bar{p}_{3,t} + b/a} = \frac{\bar{p}_{3,s} - \delta_{3,s}}{\bar{p}_{3,s} + b/a}$$

must hold. This contradicts $\bar{p}_{3,t} < \bar{p}_{3,s}$ and $\bar{q}_{3,t} > \bar{q}_{3,s}$.

# 3 Extensions

This section introduces the following extensions of the revealed preference tests: (i) additional assumptions regarding the demand function introduced by Carvajal et al. (2014), (ii) observable cost shifters as discussed in Carvajal et al. (2014), (iii) collusive price competition, which can also work as an alternative hypothesis for the above tests, (iv) cost functions that are not additively separable for different products.

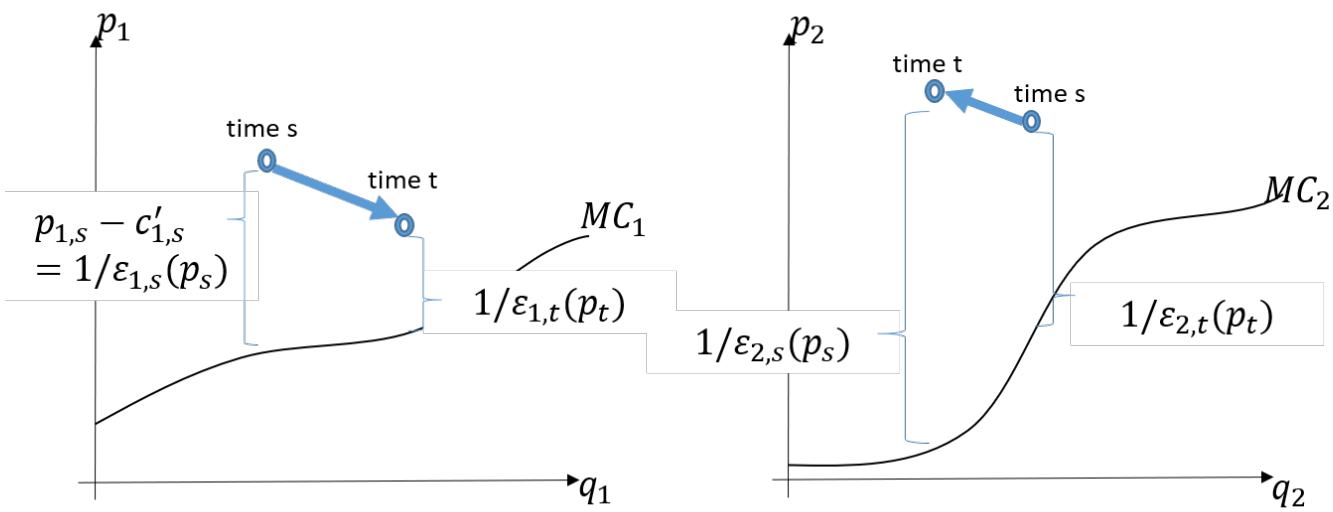
## 3.1 Additional restrictions on demand

Even though the above results provide some testable restrictions, the general model is not falsifiable. We can obtain a stricter constraint by combining the demand assumption introduced by Carvajal et al. (2014).

In order to define the additional restrictions, I first introduce some notations. I denote $\epsilon_{jt}(p) : R_+^J \to R$ as the relative decrease in the demand of good $j$ at time $t$ in response to an infinitesimal increase in its price. That is, given the demand function $Q_{jt}$ for good $j$ at time $t$ ,

$$\epsilon_{jt}(p) = -\frac{\partial Q_{j,t}(p_j,\, p_{-j})}{\partial p_j} \frac{1}{Q_{jt}(p)}$$

Figure 2: Example: Rejection by Co-evolving Property




Therefore, the own price elasticity is expressed as $p_j \epsilon_{jt}(p)$.

We then define the following properties of the demand functions.

**Definition:** A system of demand functions satisfies the *co-evolving* property if, for any $s$ and $t \in \mathcal{T}$, either

$$\epsilon_{js}(p) \geq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J}, \text{ or} \tag{12}$$

$$\epsilon_{js}(p) \leq \epsilon_{jt}(p) \text{ for all } p \in R_+^J \text{ and all } j \in \mathcal{J} \tag{13}$$

The co-evolving demand property captures the idea of *common demand shock* in Carvajal et al. (2013), which is a key component to obtain non-trivial data restrictions in their work. As seen in the above equations, if the relative slope of demand is higher for firm $j$ in market $t$, then so are the relative slopes for other firms $k \neq j$. That is, we can construct a well-defined order of demands over $\mathcal{T}$, which is common to all firms according to the relative slopes.

**Example 4. Co-evolving Property**

The power of the co-evolving property is emphasized by two products produced by different firms. Consider the same prices and quantities as the previous example, but the two goods are produced by different firms: $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\mathcal{J}_1 = \{1\}$, $\mathcal{J}_2 = \{2\}$ , $\bar{p}_{1,s} > \bar{p}_{1,t}$, $\bar{p}_{2,s} < \bar{p}_{2,t}$, $\bar{q}_{1,s} < \bar{q}_{1,t}$, and $\bar{q}_{2,s} > \bar{q}_{2,t}$ (see Fig. 2).[12] Since the two goods are produced by different firms, eq.(7) is no longer satisfied. However, the co-evolving property gives us an alternative restriction even under the general discrete/continuous model (instead

[12]The same logic is applied to multi-product firms simply by ignoring other products.

of the logit demand function). If they are single-product firms, the first-order condition is re-written as follows: $p_j - C'_j(Q_{j,t}(p)) = 1/\epsilon_{j,t}(p)$. Since the marginal costs are increasing, we can obtain an inequality of the profit margins: $1/\epsilon_{1,s}(\bar{p}_s) = \bar{p}_{1,s} - C'_1(Q_{1,s}(\bar{p}_s)) > \bar{p}_{1,t} - C'_1(Q_{1,t}(\bar{p}_t)) = 1/\epsilon_{1,t}(\bar{p}_t)$ for firm 1. Similarly, we also have $1/\epsilon_{2,s}(\bar{p}_s) < 1/\epsilon_{2,t}(\bar{p}_t)$. Therefore, the data imply $\epsilon_{1,s}(\bar{p}_s) < \epsilon_{1,t}(\bar{p}_t)$ and $\epsilon_{2,s}(\bar{p}_s) > \epsilon_{2,t}(\bar{p}_t)$. Assuming that $\epsilon_{j,s}(\cdot)$ is non-decreasing in the own price and decreasing in the other's price, we have $\epsilon_{1,s}(p) < \epsilon_{1,t}(p)$ but $\epsilon_{2,s}(p) > \epsilon_{2,t}(p)$, which contradicts the co-evolving property. In the following, I describe $\epsilon_{j,t}(\cdot)$ that is non-decreasing in the own price as *log-concave,* following Carvajal et al. (2014).[13]

Before stating the proposition, I demonstrate that the discrete/continuous model can incorporate the co-evolving property without any conflicts. For example, given multinomial logit demand, the co-evolving property is satisfied when $v_{jt}$ and $v_{kt}$ move almost in parallel over time. Since the logit demand function requires that $\epsilon_{jt}(p) = \alpha - \alpha Q_{j,t}(p)/M_t$, $\epsilon_{jt}(p) \leq \epsilon_{js}(p)$ holds if and only if $Q_{j,t}(p)/M_t \geq Q_{j,s}(p)/M_s$ holds. The co-evolving property under the logit demand function requires that $Q_{j,t}(p)/M_t \geq Q_{j,s}(p)/M_s$ if and only if $Q_{k,t}(p)/M_t \geq Q_{k,s}(p)/M_s$. This can be satisfied when $v_{jt}$ and $v_{kt}$ move almost in parallel over time or a change in $M_t$ is dominant. Log-concavity (of $Q_{j,t}(p)$) is also satisfied if $-h''_j(p_j)/h'_j(p_j)$ is non-decreasing in $p_j$. In the following proposition, I combine the discrete/continuous choice model and the co-evolving property to derive a set of necessary conditions for the data to be rationalized by price competition.

**Proposition 4.** *The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand rationalizable under convex cost functions and discrete/continuous demand functions with log-concavity and co-evolving property only if there is a permutation of $\mathcal{T}$, denoted by the function $\sigma : \mathcal{T} \to \mathcal{T}$, and real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and $m_t$ for all $s, t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$ ;
2. $0 = m_t - \{\bar{p}_j - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_k - \delta_{j,t}\} \bar{q}_{k,t}$;
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$; *and*
4. *if* $\bar{p}_{j,t} \leq \bar{p}_{j,s}$, $\bar{p}_{-j,t} \geq \bar{p}_{-j,s}$ *and* $\sigma(t) < \sigma(s)$, *then* $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$

See the appendix for the proof.

[13]Carvajal et al. (2014) impose another condition, *substitute*s *condition*, that $\epsilon_{j,t}(\cdot)$ is decreasing in others' prices. Under discrete/continuous demand, however, this condition is always satisfied.

The last condition arises from the co-evolving property and log-concavity, which characterize the common order of $\epsilon_{jt}(p)$ over time. Under the discrete/continuous demand model,

$$\epsilon_{jt}(\bar{p}_t) = \frac{h''_{j,t}(\bar{p}_{j,t})}{-h'_{j,t}(\bar{p}_{j,t})} - m_t^{-1} Q_{j,t}(\bar{p}_t) = \alpha_{j,t} - m_t^{-1} \bar{q}_{j,t} .$$

The permutation, $\sigma$, is constructed to provide a common order for $\epsilon_{jt}(p)$ (if one exists). Notably, the co-evolving property is defined by comparing $\epsilon_{jt}(p)$ and $\epsilon_{js}(p)$ for all $p$, but we only observe values corresponding to $\epsilon_{jt}(\bar{p}_t)$ and $\epsilon_{js}(\bar{p}_s)$, where $\bar{p}_t$ and $\bar{p}_s$ can take different values. To address this subtlety, the inequalities "$\bar{p}_{j,t} \le \bar{p}_{j,s}$, $\bar{p}_{-j,t} \ge \bar{p}_{-j,s}$" are added to the last condition.[14]

## 3.2 Observed cost shock

One of the important assumptions in the above tests is the time-invariant cost function. However, in reality, the cost functions shift over time, such as because of a change in input prices. We can accommodate such a shift in the revealed preference tests if cost shifters are observed.

Now, assume the following cost functions: $C_j(q_j, w_j)$ where $\partial C_j(q_j, w_j)/\partial q_j$ is increasing both in $q_j$ and $w_j$. Assume also that we observe the cost shifter $w$ in addition to price and quantity. Denote the observed price, quantity, and cost shifter as follows: $\{\bar{p}, \bar{q}, \bar{w}\}$ where $\bar{x} = (\bar{x}'_1, ..., \bar{x}'_T)'$ and $\bar{x}_t = (\bar{x}_{1,t}, ..., \bar{x}_{J,t})'$ for $x = p, q, w$. Then, the restriction is modified as follows.

*Remark* 1. The set of observations $\{\bar{p}, \bar{q}, \bar{w}\}$ is Bertrand-rationalizable under marginal cost functions increasing in own quantity and a cost shifter and discrete/continuous demand functions only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and $m_t$ for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; and
3. $\delta_{j,t'} \ge \delta_{j,t}$ whenever $(\bar{q}_{j,t'}, \bar{w}_{j,t'}) > (\bar{q}_{j,t}, \bar{w}_{j,t})$.

In the above claim, I use a partial order for the third condition since, if $\bar{q}_{j,t'} > \bar{q}_{j,t}$ and $\bar{w}_{j,t'} < \bar{w}_{j,t}$, then we cannot tell which marginal cost is higher. Tests for price competition under logit, CES, and HARA $h$, can be also derived analogously.

[14] For this proposition, I provide only the necessity of the conditions. For the proof of sufficiency, I need to reconstruct the demand functions satisfying both a discrete/continuous structure and the co-evolving property from any parameters satisfying conditions 1-4.

## 3.3 Collusive price competition

This section discusses revealed preference tests of collusive price competition. Each firm is assumed to choose their own price while (partially) internalizing the effect on the other firms as in Miller and Weinberg (2017) and Sullivan (2016). More specifically, firm $f$ maximizes an objective function

$$\pi_{f,t}(p) = \sum_{f' \in \mathcal{F}} \left[ \phi_{f,f'} \sum_{k \in J_{f'}} \left\{ Q_{k,t}(p)\, p_k - C_k\left(Q_{k,t}(p)\right) \right\} \right]$$

given the others' prices at time $t$, where $\phi_{f,f'} \in [0,\, 1]$ is firm $f$'s weight on firm $f'$'s profit. The first-order condition w.r.t. $p_j$ is written as follows:

$$0 \;=\; Q_{j,t}(p) + \sum_{f' \in \mathcal{F}} \left[ \phi_{f,f'} \sum_{k \in J_{f'}} \left\{ p_k - C_k'\left(Q_{k,t}(p)\right) \right\} \frac{\partial Q_{k,t}(p)}{\partial p_j} \right].$$

This first-order condition is simplified using the derivatives of discrete/continuous demand functions and dividing both sides by the market share of product $j$. Then, the first-order condition gives the following data restriction:

$$0 = m_t - \left\{ \bar{p}_{j,t} - \delta_{j,t} \right\} m_t \alpha_{j,t} + \sum_{f' \in \mathcal{F}} \phi_{f,f'} \sum_{k \in J_{f'}} \left\{ \bar{p}_{k,t} - \delta_{k,t} \right\} \bar{q}_{k,t}.$$

Then, we state a modified version of the revealed preference test as follows:

*Remark* 2. The set of observations $\{\bar{p},\, \bar{q}\}$ is rationalized by collusive price competition under convex cost functions and logit demand functions *only if* there exist real numbers $\alpha_t$, $\delta_{j,t}$, and $m_t$, for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ and $\phi_{f,f'}$ for any $f,\, f' \in \mathcal{F}$ such that the following hold:

1. $\alpha_t > 0$, $\delta_{j,t} > 0$, $m_t > 0$, $\phi_{f,f'} \in [0,1]$ ;
2. $0 = m_t - \left\{ \bar{p}_{j,t} - \delta_{j,t} \right\} m_t \alpha_t + \sum_{f' \in \mathcal{F}} \phi_{f,f'} \sum_{k \in J_{f'}} \left\{ \bar{p}_{k,t} - \delta_{k,t} \right\} \bar{q}_{k,t}$; and
3. $0 \le \left( \delta_{j,t'} - \delta_{j,t} \right) \left( \bar{q}_{j,t'} - \bar{q}_{j,t} \right)$.

In the above test, the demand is assumed to be a logit demand funcion since any data are rationalized by the general discrete/continuous model, and the additional parameter $\phi_{f,f'}$ would weakly loosen the restrictions. In contrast, under a logit demand function, the common markup property still holds, so the data shown in Example 2 are not rationalized by the collusive price competition.

Now, a natural question is to what extent the additional parameter $\phi_{f,f'}$ loosens the data restrictions. In fact, it loosens the data restrictions less than might be expected. For example, consider the following variant of the above test. Suppose that there are two firms $a$ and $b$, producing products $a$ and $b$, respectively, and one attempts to test whether a set of observed data $\{\bar{p}, \bar{q}\}$ is rationalized by collusive price competition under a logit demand function. If the firms are competing in prices, the data $\{\bar{p}, \bar{q}\}$ must satisfy the following conditions for some $\alpha_t$, $\delta_{j,t}$, $m_t$, and $\phi_{f,f'}$:

$$\begin{aligned} 0 &= m_t - \{\bar{p}_{a,t} - \delta_{a,t}\} m_t\alpha_t + \{\bar{p}_{a,t} - \delta_{a,t}\} \bar{q}_{a,t} + \phi_{a,b} \{\bar{p}_{b,t} - \delta_{b,t}\} \bar{q}_{b,t} \\ 0 &= m_t - \{\bar{p}_{b,t} - \delta_{b,t}\} m_t\alpha_t + \{\bar{p}_{b,t} - \delta_{b,t}\} \bar{q}_{b,t} + \phi_{b,a} \{\bar{p}_{a,t} - \delta_{a,t}\} \bar{q}_{a,t} \end{aligned} \tag{14}$$

This implies the following condition:

$$\begin{aligned} \{\bar{p}_{a,t} - \delta_{a,t}\} \{m_t\alpha_t - (1-\phi_{b,a}) \bar{q}_{a,t}\} &= \{\bar{p}_{b,t} - \delta_{b,t}\} \{m_t\alpha_t - (1-\phi_{a,b}) \bar{q}_{b,t}\} \\ \Leftrightarrow \frac{\bar{p}_{a,t} - \delta_{a,t}}{\bar{p}_{b,t} - \delta_{b,t}} &= \frac{m_t\alpha_t - (1-\phi_{a,b}) \bar{q}_{b,t}}{m_t\alpha_t - (1-\phi_{b,a}) \bar{q}_{a,t}} \end{aligned} \tag{15}$$

Note that for any $m_t\alpha_t$, $\delta_{j,t}$, and $\phi_{f,f'}$ satisfying the above equation, we can determine the corresponding $m_t$ and $\alpha_t$ from the original first-order equation, eq. (14). Therefore, constraints characterized by eq. (15) are equivalent to constraints characterized by eq. (14).

Now, focusing on symmetric $\phi_{f,f'}$'s, i.e., $\phi_{a,b} = \phi_{b,a} \equiv \phi$, as is often the case, data restrictions for price competition, $\phi = 0$, and restrictions for collusive price competition, $\phi \in [0,1)$, are equivalent. That is, if there exists a set of $\left\{\{\alpha_t\}_t, \{m_t\}_t, \{\delta_{j,t}\}_{j,t}, \phi\right\}$ for $\phi \in [0,1)$ satisfying eq. (15), we can also find $\left\{\{\widehat{\alpha}_t\}_t, \{\widehat{m}_t\}_t, \{\delta_{j,t}\}_{j,t}\right\}$ combined with $\phi = 0$ satisfying eq. (15).

The construction of $\{\widehat{\alpha}_t\}_t$ and $\{\widehat{m}_t\}_t$ is as follows. Suppose that $\left\{\{\alpha_t\}_t, \{m_t\}_t, \{\delta_{j,t}\}_{j,t}, \phi\right\}$ satisfies eq. (15):

$$\frac{\bar{p}_{a,t} - \delta_{a,t}}{\bar{p}_{b,t} - \delta_{b,t}} = \frac{m_t\alpha_t - (1-\phi) \bar{q}_{b,t}}{m_t\alpha_t - (1-\phi) \bar{q}_{a,t}} \text{ for any } t$$

Now, observe that $\left\{\{\widehat{m\alpha}_t\}_t, \{\delta_{j,t}\}_{j,t}\right\}$ such that $\widehat{m\alpha}_t = m_t\alpha_t / (1-\phi)$ satisfies

$$\frac{\bar{p}_{a,t} - \delta_{a,t}}{\bar{p}_{b,t} - \delta_{b,t}} = \frac{\widehat{m\alpha}_t - \bar{q}_{b,t}}{\widehat{m\alpha}_t - \bar{q}_{a,t}} \text{ for any } t.$$

Furthermore, we can construct the corresponding $\widehat{m}_t$ and $\widehat{\alpha}_t$ as

$$\widehat{m}_t \equiv \{\bar{p}_{a,t} - \delta_{a,t}\} \widehat{m\alpha}_t - \{\bar{p}_{a,t} - \delta_{a,t}\} \bar{q}_{a,t}$$

and $\widehat{\alpha}_t \equiv \widehat{m\alpha}_t / \widehat{m}_t$. Thus, the data can also be explained by the price competition ($\phi = 0$).

It is also worth noting that asymmetric $\phi_{f,\,f'} \in [0,\, 1]$ generates strictly loose conditions versus symmetric $\phi \in [0, 1]$.

We can test a perfect collusion model as a special case of the above specification, where $\phi_{f,\,f'} = 1$ for all $f$ and $f'$. It is mathematically the same model as the baseline model but with a different definition for a firm. Therefore, we obtain the same result as in Section 2 with a different definition for the firm. To be more specific, I write an immediate corollary of Theorem 1 (characterization of the test) and Corollary 1 (unfalsifiability) for perfect collusion under a general discrete/continuous demand model. In the following, I call $\{\bar{p},\, \bar{q}\}$ collusion-rationalizable if there exist some demand and cost functions under which $\{\bar{p},\, \bar{q}\}$ is generated as a result of profit maximization of a collusive group of firms.

**Corollary 2.** *The set of observations $\{\bar{p},\, \bar{q}\}$ is collusion-rationalizable under convex cost functions and discrete/continuous demand functions if and only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and $m_t$ for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t} > 0,\ \delta_{j,t} > 0,\ m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\}\, m_t \alpha_{j,t} + \sum_{f' \in \mathcal{F}} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\}\, \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})\,(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

*Furthermore, any data, $\{\bar{p},\, \bar{q}\}$, are collusion-rationalizable under convex cost functions and discrete/continuous demand functions.*

By restricting a class of demand functions to the logit, we can also obtain a falsifiable model, which is summarized as an immediate corollary of Proposition 1.

**Corollary 3.** *(Logit) The set of observations $\{\bar{p},\, \bar{q}\}$ is collusion-rationalizable under convex cost functions and logit demand functions if and only if there exist real numbers $\alpha_t$, $\delta_{j,t}$, and $m_t$ for all $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha > 0,\ \delta_{j,t} > 0,\ m_t > 0$ ;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\}\, m_t \alpha_t + \sum_{k \in J_f} \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\}\, \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})\,(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

It is worth noting that a perfect collusion ($\phi = 1$) is strictly more restrictive than a collusive price competition ($\phi \in (0,\, 1)$) or a price competition $\phi = 0$ because the common mark-up property applies for all products in the market under the perfect collusion case but not for(collusive) price competition.

**Example 5. Price Competition v.s. Collusion**

For example, suppose that we observe the following data on products 1 and 2, which are produced by firm 1 and 2, respectively: $(\bar{p}_{j,\tau}, \bar{q}_{j,\tau})_{j=1,2,\,\tau=s,t}$ s.t. $\bar{p}_{1,t} = \bar{p}_{1,s} = \bar{p}_{2,t} = 1$, $\bar{p}_{2,s} = 0.95$ , $\bar{q}_{1,t} = \bar{q}_{1,s} = 1$, $\bar{q}_{2,t} = 2$, $\bar{q}_{2,s} = 3$. Then, such data can be rationalized by a Bertrand competition with convex cost functions such that $C_1'(1) = 0.6$, $C_2'(2) = 0.2$, $C_2'(3) = 0.3$, and market sizes for each time $M_t = 4$ and $M_s = 11$, which correspond to $m_t = 0.12$, $m_s = 4$, $\alpha_t = 100/3$, and $\alpha_s = 11/4$.[15] Note that the example here is not a special case of Example 1 in Section 2.1 because quantities of product 2 are now always greater than the quantity of product 1, while an inequality $\bar{q}_{2,t} > \bar{q}_{1,t} = \bar{q}_{1,s} > \bar{q}_{2,s}$ holds in Example 1. In contrast, the data cannot be rationalized by perfect collusion because the data contradict the common mark-up property between product 1 and 2, which should hold under perfect collusion.

## 3.4 Non-separable cost functions

In this part, I further generalize the model to price competition with convex cost functions that are not additively separable. Each firm $f \in \{1, ..., F\}$ produces a set of products $\mathcal{J}_f \subset \mathcal{J}$ s.t. $\mathcal{J}_f \cap \mathcal{J}_g = \phi$ for $f \neq g$ and denote $J_f = |\mathcal{J}_f|$. At the same time, firm $f \in \{1, ..., F\}$ faces a cost function, $C_f : R_+^{J_f} \to R$ , which is assumed to be convex and twice continuously differentiable. Denote a vector of demand functions for firm $f$ as $Q_{f,t}(p) = [Q_{j,t}(p)]_{j\in\mathcal{J}_f}$ Then, the profit function for firm $f$ at time $t$ becomes $\pi_{f,t}(p) = \sum_{j\in\mathcal{J}_f}\{Q_{j,t}(p)\,p_j\} - C_f(Q_{f,t}(p))$ and the first-order condition w.r.t. $p_j$ becomes

$$0 \;=\; Q_{j,t}(p) + \sum_{k\in J_f}\left\{p_k - \frac{\partial C(Q_{f,t}(p))}{\partial Q_{k,t}}\right\}\frac{\partial Q_{k,t}(p)}{\partial p_j}.$$

The only difference in the first-order condition from that with additively separable cost functions is that $\partial C(Q_{f,t}(p))/\partial Q_{k,t}$ replaces $C_k'(Q_{k,t}(p))$ (see eq. (1)). Therefore, we have to find $\delta_{j\,t}$, which now corresponds to $\partial C(Q_{f,t}(p))/\partial Q_{k,t}$ instead of $C_k'(Q_{k,t}(p))$. The transformation of the first-order conditions using the discrete/continuous demand structure is applied as in Section 2. The convexity of the cost function implies

$$0 \le (\nabla C_f(q_{f,t}) - \nabla C_f(q_{f,s}))'(q_{f,t} - q_{f,s}).$$

[15] Like eq. (15), the first-order conditions can be reduced to

$$\frac{1 - C_1'(1)}{1 - C_2'(2)} = \frac{M_t - 2}{M_t - 1} \text{ and } \frac{M_s - 3}{M_s - 1} = \frac{1 - C_1'(1)}{0.95 - C_2'(3)},$$

which are easier to verify than the original first-order conditions.

Combining these conditions, we obtain a test for price competition with non-separable and convex cost functions.

**Theorem 2.** *(Discrete/Continuous): The set of observations $\{\bar{p}, \bar{q}\}$ is Bertrand-rationalizable under a non-separable and convex cost function and a discrete/continuous demand function if and only if there exist real numbers $\alpha_{j,t}$, $\delta_{j,t}$, and $m_t$ for any $t \in \mathcal{T}$ and $j \in \mathcal{J}$ such that the following hold:*

1. $\alpha_{j,t} > 0$, $\delta_{j,t} > 0$, $m_t > 0$;
2. $0 = m_t - \{\bar{p}_{j,t} - \delta_{j,t}\} m_t \alpha_{j,t} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{k,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{f,t'} - \delta_{f,t})' (\bar{q}_{f,t'} - \bar{q}_{f,t})$.

The only difference in the proof is the reconstruction of the cost functions and the uniqueness of the equilibrium. First, I construct a preliminary cost function, $\widetilde{C}_f(q_f) = \max_t \{V_{f,t} + \delta_{f,t} \cdot q_f\}$ for some $V_{f,t}$ such that $\widetilde{C}_f(\bar{q}_{f,t}) = V_{f,t} + \delta_{f,t} \cdot \bar{q}_{f,t}$ (i.e., $\nabla \widetilde{C}_f(\bar{q}_{f,t}) = \delta_{f,t}$). This cost function is convex, but not differentiable everywhere. We can construct a convex and differentiable cost function by smoothing this. For example, we can use a convolution-based smoothing technique:

$$\bar{C}_f(q_f) = \int \widetilde{C}_f(q_f - z) \mu_\epsilon(z) dz$$

where $\mu_\epsilon(z)$ is a density function for a uniform distribution around 0 with radius $\epsilon > 0$. For a sufficiently small $\epsilon > 0$, we make the reconstructed cost function convex and differentiable while keeping $\nabla \bar{C}_f(\bar{q}_{f,t}) = \delta_{f,t}$.

For the uniqueness of the equilibrium, most arguments in the proof hold by replacing $C'_j(\bar{Q}_{j,t}(p))$ with $\partial C_f(\bar{Q}_{f,t}(p)) / \partial q_j$. The remaining task is to prove that the derivative of $[\nu_j(p)]_j$ is negative definite where

$$\nu_j(p) = \left\{ p_j - \frac{\partial C_f}{\partial q_j} \left( \bar{Q}_f(\boldsymbol{p}) \right) \right\} \alpha_j.$$

The partial derivatives are

$$\begin{aligned} \frac{\partial \nu_j(\mathbf{p})}{\partial p_j} &= \alpha_j \left( 1 - \sum_{l \in \mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_j} \frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_j} \right) \\ \frac{\partial \nu_j(\mathbf{p})}{\partial p_k} &= -\alpha_j \sum_{l \in \mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_j} \frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_k} \end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'} &= \Lambda\left\{I - \begin{bmatrix} \sum_{l\in\mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_1}\frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_1} & \sum_{l\in\mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_1}\frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_n} \\ \sum_{l\in\mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_n}\frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_1} & \sum_{l\in\mathcal{J}_f} \frac{\partial^2 C_f}{\partial q_l \partial q_n}\frac{\partial \bar{Q}_l(\mathbf{p})}{\partial p_n}\end{bmatrix}\right\} \\ &= \Lambda\left\{I - \Gamma(\mathbf{p})\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}\right\}\end{aligned}$$

where

$$\Gamma(\mathbf{p}) = \begin{bmatrix} \frac{\partial^2 C_f}{\partial q_1 \partial q_1} & \cdots & \frac{\partial^2 C_f}{\partial q_1 \partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 C_f}{\partial q_n \partial q_1} & \cdots & \frac{\partial^2 C_f}{\partial q_n \partial q_n}\end{bmatrix}.$$

Note that $\Gamma(\mathbf{p})$ is positive semi-definite because $C_f$ is convex. Now, the same algebra as Theorem 1 applies by re-interpreting $\Gamma(\mathbf{p})$ in the proof of Theorem 1.

# 4 Implementation

The existence of parameters satisfying the inequality constraints can be checked by minimizing a loss function over a set of parameters, given the data observed, and checking whether the minimized value is close to zero. For instance, for an inequality $g(\theta; \bar{p}, \bar{q}) \geq 0$, we can construct a loss function $(\min\{0,\, g(\theta; \bar{p}, \bar{q})\})^2$. Similarly, for a vector of equality constraints $\boldsymbol{h}(\theta; \bar{p}, \bar{q}) = \mathbf{0}$ and a vector of inequality constraints $\boldsymbol{g}(\theta; \bar{p}, \bar{q}) \geq \mathbf{0}$, we can construct a loss function

$$\boldsymbol{h}(\theta; \bar{p}, \bar{q})^T \boldsymbol{h}(\theta; \bar{p}, \bar{q}) + \tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q})^T \tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q}),$$

where $\tilde{\boldsymbol{g}}(\theta; \bar{p}, \bar{q}) = [\min\{0,\, g_i(\theta; \bar{p}, \bar{q})\}]_i$. In general, this minimization faces computational issues similar to those of estimation with moment inequalities.[16] With logit demand and a slightly modified data requirement, the inequalities are written as linear constraints on parameters so that we can check the existence of parameters using off-the-shelf tools for linear constraints. In the following, I assume that the market size $\{\bar{M}_t\}_t$, prices $\{\bar{p}_{j,t}\}$, and quantities $\{\bar{q}_{j,t}\}$ are observable, as is always the case when the logit demand function can be estimated. The market shares of products at each time $\{\bar{s}_{j,t}\}$ are also observable since $\bar{s}_{j,t} = \frac{\bar{q}_{j,t}}{\bar{M}_t}$. Then, considering that $m = \frac{M}{\alpha}$ under logit demand, and with the replacement of $\frac{1}{\alpha} = \tilde{\alpha}$, the data restrictions for price competition under the logit demand function are characterized by a set of linear constraints on parameters $\tilde{\alpha}_{j,t}$, and $\delta_{j,t}$.

[16]The loss function tends to have a basin at the bottom with kinks around it. Therefore, standard optimization algorithms do not work well.

**Corollary 4.** *(Logit) The set of observations* $\{\bar{p}, \bar{q}, \bar{M}\}$ *is Bertrand-rationalizable under convex cost functions and logit demand functions if and only if there exist real numbers* $\tilde{\alpha}_t$, $\delta_{j,t}$, *and* $m_t$ *for all* $t \in \mathcal{T}$ *and* $j \in \mathcal{J}$ *such that the following hold:*

1. $\tilde{\alpha}_t > 0$, $\delta_{j,t} > 0$;
2. $0 = \bar{M}_t \tilde{\alpha}_t - \{\bar{p}_{j,t} - \delta_{j,t}\} + \sum_{k \in J_f} \{\bar{p}_{k,t} - \delta_{j,t}\} \bar{q}_{k,t}$; *and*
3. $0 \leq (\delta_{j,t'} - \delta_{j,t})(\bar{q}_{j,t'} - \bar{q}_{j,t})$.

*Proof.* The proof is an immediate corollary of Proposition 1. □

Thus, we can use standard algorithms for linear constraints to check the constraint.

# 5 Summary

In this paper, I modify a Bertrand assumption test introduced by Carvajal et al. (2014) to allow it to be implemented for multi-product firms. To address difficulties caused by cannibalization effects, I employ the discrete/continuous demand function introduced by Nocke and Schutz (2018), which includes the multinomial logit demand function and the CES demand function as special cases. In the main theorem, I provide the necessary and sufficient condition for data to be rationalized by Bertrand competition among multi-product firms under the discrete/continuous model. The test is implementable without any IVs, and rejection by it deterministically implies misspecification of the model rather than a peculiar realization of structural error terms. Under the general discrete/continuous model, any data would satisfy the necessary and sufficient condition to be rationalized by price competition, while some data are not rationalized by price competition under more restrictive demand specifications such as the logit demand function, CES demand function, or a discrete/continuous demand function with HARA $h$. I also discuss additional restrictions on the demand function discussed in previous research, a test with observed cost shifters, a test for collusive price competition, and a simple implementation of the logit demand function. The tests can also be applied for price competition, collusive price competition, and collusion with cost functions that are not additively separable.

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# Appendix

**Proof of Theorem 1.** For sufficiency, we only need to construct cost and demand functions for each firm whose profit function is maximized at $\bar{p}_{j,t}$, $\bar{q}_{j,t}$.

First, consider a reconstruction of demand function. If the data satisfy the restriction defined in Theorem 1, we should be able to find $\alpha_{j,t}$ which corresponds to $\frac{h''_{j,t}(\bar{p}_{j,t})}{-h'_{j,t}(\bar{p}_{j,t})}$ for any $j$ and $t$, where $h_{j,t} : R_+ \to R$ represents the true data-generating process. For reconstruction of demand functions, I consider $\bar{h}_{j,t} : R_+ \to R$ s.t. $\frac{\bar{h}''_{j,t}(p_j)}{-\bar{h}'_{j,t}(p_j)} = \alpha_{j,t}$ for any $p_j \in R_+$. Note that this condition holds for any $p_j \in R_+$, but not only for $\bar{p}_{j,t}$. This is analogous to the construction of utility function in Afriat (1967), where a reconstructed utility function is locally linear even though the data-generating demand function can be non-linear. Since the constant $\frac{\bar{h}''_{j,t}(p_j)}{\bar{h}'_{j,t}(p_j)}$ imply that $\bar{h}_{j,t}(p_j)$ can be represented as CARA function with risk averseness $\alpha_{j,t}$, we can represent $\bar{h}_{j,t}(p_j) = \exp\{v_{jt} - \alpha_{jt}p_j\}$ for some $v_{jt}$. Then, we can construct a demand function,

$$\bar{Q}_{j,t}(p) = m_t \frac{-\bar{h}'_{j,t}(p_j)}{\bar{h}_{0,t} + \sum_k \bar{h}_{k,t}(p_k)} = m_t \frac{\alpha_{j,t}\exp\{v_{j,t} - \alpha_{j,t}p_j\}}{\bar{h}_{0,t} + \sum_k \exp\{v_{k,t} - \alpha_{k,t}p_k\}}$$

for some $\bar{h}_{0,t}$. Here, I denote the reconstructed demand function as $\bar{Q}_{j,t}(p)$ in order to distinguish it from the demand function in the true data-generating process, $Q_{j,t}(p)$. Now, $v_{jt}$ can be chosen to satisfy a system of $K$ equations,

$$m_t \frac{\alpha_{j,t}\exp\{v_{j,t} - \alpha_{j,t}\bar{p}_{j,t}\}}{\bar{h}_{0,t} + \sum_k \exp\{v_{k,t} - \alpha_{k,t}\bar{p}_{k,t}\}} = \bar{q}_{jt}$$

for all $j$, similar to an inversion of share functions in logit specifications, discussed in Berry (1994).

Since $(\delta_{jt}, q_{jt})$ satisfies the co-monotone property, we can use monotone cubic interpolation to reconstruct the increasing and continuously differentiable $\bar{C}'(\cdot)$. Then, we can reconstruct $\bar{C}(q) = \int_0^q \bar{C}'(x)\,dx$, which is convex and twice continuously differentiable.[17]

The final step is to prove that $(\bar{p}, \bar{q})$ is an equilibrium under the reconstructed demand and cost functions. Since the reconstructed profit function is continuously differentiable, first-order conditions must be satisfied at the optimal price. Therefore, we only need to show that a solution to the first-order conditions for each firm is unique given the other firms' strategies. To do so, I use the common $\iota$-markup property examined in Nocke and Schutz (2018). The following part is closely related to the proofs in Nocke and Schutz (2018)

[17] Carvajal et al. (2013, 2014) reconstruct the cost function as an upper envelope of linear cost functions, whose slope is determined by $\delta_{j,t}$'s. Instead, in this paper, I use cubic interpolation for differentiability, which is necessary for the inversion of $\iota$-markup.

(especially Lemma F), despite a few differences. First, we do not need to prove the existence of an equilibrium since we already have data as an equilibrium candidate. Therefore, we only need to show that those data are an equilibrium. Second, we consider a more general cost specification than Nocke and Schutz (2018). It complicates the inversion from $\mu^f$ to price vectors since marginal cost is not a constant, but a function of product quantity. Third, the reconstructed demand function is a special case of the demand function in Nocke and Schutz (2018). Therefore, we can circumvent the difficulty arising from a general cost function by specifying the shape of the demand function.

In the following, I omit the subscript for time $t$ considering that the static NE is repeated and the following logic is applied for each $t$. Then, I denote the reconstructed demand function as $\bar{Q}_j(p) = m\frac{-\bar{h}'_j(p_j)}{\bar{h}_0+\sum_k \bar{h}_k(p_k)} = m\frac{\alpha_j \exp\{v_j-\alpha_j p_j\}}{\bar{h}_0+\sum_k \exp\{v_k-\alpha_k p_k\}}$ and $\bar{h}'_j(p_j) = -\alpha_j \exp\{v_j - \alpha_j p_j\}$, $\bar{h}''_j(p_j) = \alpha_j^2 \exp\{v_j - \alpha_j p_j\}$, and $\frac{\bar{h}''_j(p_j)}{-\bar{h}'_j(p_j)} = \alpha_j$. Since we now consider a maximization problem of a specific firm given the other firm's strategy, let us denote $\bar{h}_0 + \sum_{k \notin J_f} \bar{h}_k(p_k) = H_0$ and $J_f = \{1, ..., n\}$ without loss of generality. By the FOC, we have the following for any $j \in J_f$

$$\left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\mathbf{p})\right)\right\} \frac{\bar{h}''_j(p_j)}{-\bar{h}'_j(p_j)} = 1 + m^{-1} \sum_{k \in J_f} \left\{p_k - \bar{C}'_k\left(\bar{Q}_k(\mathbf{p})\right)\right\} \bar{Q}_k(\mathbf{p}) \tag{16}$$

Since the RHS is same for any $j \in J_f$, the solution of a system of equations defined by (16) for any $j \in J_f$ satisfies

$$\nu_j(\mathbf{p}) \equiv \left\{p_j - \bar{C}'_j\left(\bar{Q}_j(\boldsymbol{p})\right)\right\} \alpha_j = \mu^f$$

for any $j \in J_f$. Let $\nu(\mathbf{p}) = [\nu_1(\mathbf{p}), ..., \nu_n(\mathbf{p})]'$. Then, $\mathbf{p} = \nu^{-1}\left(\mathbf{1}\mu^f\right) \equiv r\left(\mu^f\right) \equiv \left[r_1\left(\mu^f\right), ..., r_n\left(\mu^f\right)\right]'$ at the solution of (16). Then, we can rewrite the condition (16) as

$$\begin{aligned}
\mu^f &= 1 + m^{-1} \sum_{k \in J_f} \left\{r_k\left(\mu^f\right) - \bar{C}'_k\left(\bar{Q}_k\left(r\left(\mu^f\right)\right)\right)\right\} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1} \sum_{k \in J_f} \underbrace{\left\{r_k\left(\mu^f\right) - \bar{C}'_k\left(\bar{Q}_k\left(r\left(\mu^f\right)\right)\right)\right\} \alpha_k}_{\mu^f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
&= 1 + m^{-1}\mu^f \sum_{k \in J_f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) \\
\Leftrightarrow 0 &= 1 + \mu^f \left\{m^{-1} \sum_{k \in J_f} \frac{1}{\alpha_k} \bar{Q}_k\left(r\left(\mu^f\right)\right) - 1\right\} \equiv \psi\left(\mu^f\right)
\end{aligned}$$

Then, the uniqueness of the solution of the first-order condition is proved by the strict monotonicity of $\psi\left(\mu^f\right)$. Again, the existence of a solution can be omitted since the data satisfy the first-order condition by the construction of $\left(\bar{Q}_j(\cdot), \bar{C}_j(\cdot)\right)_{j\in J_f}$. By taking a derivative w.r.t. $\mu^f$

$$\begin{aligned}
\psi'\left(\mu^f\right) &= \underbrace{\sum_{k\in J_f}\frac{\exp\left\{v_k-\alpha_k r_k\left(\mu^f\right)\right\}}{H_0+\sum_l \exp\left\{v_l-\alpha_l r_l\left(\mu^f\right)\right\}}-1}_{<0} \\
&\quad \underbrace{+\mu^f m^{-1}\sum_{k\in J_f}\frac{1}{\alpha_k}\underbrace{\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial \mathbf{p}}|_{\mathbf{p}=r\left(\mu^f\right)}}_{1\times n}\underbrace{r'\left(\mu^f\right)}_{n\times 1}}_{\equiv A}
\end{aligned}$$

It is sufficient to show that $A\leq 0$.

$$\begin{aligned}
A &= \mu^f m^{-1}\underbrace{\left[\frac{1}{\alpha_1},...,\frac{1}{\alpha_n}\right]}_{1\times n}\underbrace{\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n\times n}\underbrace{\frac{\partial \nu^{-1}(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}\underbrace{\mathbf{1}}_{n\times 1} \\
&= \mu^f m^{-1}\underbrace{\mathbf{1}'}_{1\times n}\underbrace{\Lambda^{-1}\underbrace{\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=r\left(\mu^f\right)}}_{n\times n}\underbrace{\frac{\partial \nu^{-1}(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f}}_{n\times n}}_{\equiv B}\underbrace{\mathbf{1}}_{n\times 1}
\end{aligned}$$

where $\Lambda = \begin{bmatrix}\alpha_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n\end{bmatrix}$. Since $\mu^f > 0$ and $m > 0$, we only need to show that $B$ is negative semi-definite.

To examine the derivatives of $\nu^{-1}(\mathbf{m})$, we first consider the derivative of $\nu$. Recall that $\nu_j(\mathbf{p}) \equiv \left\{p_j - C_j'\left(\bar{Q}_j(p)\right)\right\}\alpha_j$. Then, the partial derivatives are

$$\begin{aligned}
\frac{\partial \nu_k(\mathbf{p})}{\partial p_k} &= \alpha_k\left(1-C_k''\left(\bar{Q}_k(\mathbf{p})\right)\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_k}\right) \\
\frac{\partial \nu_k(\mathbf{p})}{\partial p_j} &= -\alpha_k C_k''\left(\bar{Q}_k(\mathbf{p})\right)\frac{\partial \bar{Q}_k(\mathbf{p})}{\partial p_j}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'} &= \Lambda\left\{I-\begin{bmatrix} C_1''\left(\bar{Q}_1(\mathbf{p})\right)\frac{\partial \bar{Q}_1(\mathbf{p})}{\partial p_1} & C_1''\left(\bar{Q}_1(\mathbf{p})\right)\frac{\partial \bar{Q}_1(\mathbf{p})}{\partial p_n} \\ C_n''\left(\bar{Q}_n(\mathbf{p})\right)\frac{\partial \bar{Q}_n(\mathbf{p})}{\partial p_1} & C_n''\left(Q_n(\mathbf{p})\right)\frac{\partial \bar{Q}_n(\mathbf{p})}{\partial p_n}\end{bmatrix}\right\} \\
&= \Lambda\left\{I-\Gamma(\mathbf{p})\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}\right\}
\end{aligned}$$

where $\Gamma(\mathbf{p}) = \begin{bmatrix} C_1''\left(\bar{Q}_1(\mathbf{p})\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_n''\left(\bar{Q}_n(\mathbf{p})\right) \end{bmatrix}$. Then,

$$\begin{aligned}
B &= \Lambda^{-1}\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\frac{\partial r(\mathbf{m})}{\partial \mathbf{m}'}|_{\mathbf{m}=\mathbf{1}\mu^f} \\
&= \Lambda^{-1}\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\left[\frac{\partial \nu(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right]^{-1} \\
&= \Lambda^{-1}\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\left[\Lambda\left\{I - \Gamma(\mathbf{p})\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}\right\}\right]^{-1} \\
&= \Lambda^{-1}\left(\left(\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\left\{I - \Gamma(\mathbf{p})\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}\right\}^{-1}\Lambda^{-1} \\
&= \Lambda^{-1}\left(\left\{I - \Gamma(\mathbf{p})\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}\right\}\left(\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1} \\
&= \Lambda^{-1}\left(\left(\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1} - \Gamma(\mathbf{p})\right)^{-1}\Lambda^{-1} \\
&= -\Lambda^{-1}\left(\Gamma(\mathbf{p}) - \left(\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}|_{\mathbf{p}=\bar{\mathbf{p}}}\right)^{-1}\right)^{-1}\Lambda^{-1}
\end{aligned}$$

Now, $B$ is negative definite if $\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'}$ is negative definite;

$$\begin{aligned}
\frac{\partial \bar{Q}(\mathbf{p})}{\partial \mathbf{p}'} &= \begin{bmatrix} -m^{-1}\bar{Q}_1(p)\left\{\alpha_1 m - \bar{Q}_1(p)\right\} & m^{-1}\bar{Q}_1(p)\,Q_2(p) & \cdots & m^{-1}\bar{Q}_1(p)\,Q_n(p) \\ m^{-1}\bar{Q}_1(p)\,\bar{Q}_2(p) & -m^{-1}\bar{Q}_2(p)\left\{\alpha_2 m - \bar{Q}_2(p)\right\} & & m^{-1}\bar{Q}_2(p)\,\bar{Q}_n(p) \\ \vdots & & \ddots & \vdots \\ m^{-1}\bar{Q}_1(p)\,\bar{Q}_n(p) & m^{-1}\bar{Q}_2(p)\,\bar{Q}_n(p) & \cdots & -m^{-1}\bar{Q}_n(p)\left\{\alpha_n m - \bar{Q}_n \right. \end{bmatrix} \\
&= m^{-1}\begin{bmatrix} -\bar{Q}_1(p)\left\{\alpha_1 m - \bar{Q}_1(p)\right\} & \bar{Q}_1(p)\,\bar{Q}_2(p) & \cdots & \bar{Q}_1(p)\,\bar{Q}_n(p) \\ \bar{Q}_1(p)\,\bar{Q}_2(p) & -\bar{Q}_2(p)\left\{\alpha_2 m - \bar{Q}_2(p)\right\} & & \bar{Q}_2(p)\,\bar{Q}_n(p) \\ \vdots & & \ddots & \vdots \\ \bar{Q}_1(p)\,\bar{Q}_n(p) & \bar{Q}_2(p)\,\bar{Q}_n(p) & \cdots & -\bar{Q}_n(p)\left\{\alpha_n m - \bar{Q}_n(p)\right\} \end{bmatrix} \\
&= m^{-1}\left\{\begin{bmatrix} \bar{Q}_1(p)\,\bar{Q}_1(p) & \cdots & \bar{Q}_1(p)\,\bar{Q}_n(p) \\ \vdots & \ddots & \vdots \\ \bar{Q}_1(p)\,\bar{Q}_n(p) & \cdots & \bar{Q}_n(p)\,\bar{Q}_n(p) \end{bmatrix} - m\begin{bmatrix} \alpha_1\bar{Q}_1(p) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n\bar{Q}_n(p) \end{bmatrix}\right\}
\end{aligned}$$

Then,

$$
\begin{aligned}
x'\frac{\partial \bar{Q}\left(\mathbf{p}\right)}{\partial \mathbf{p}'}x &= m^{-1}\left\{x'\bar{Q}\left(p\right)\bar{Q}\left(p\right)'x - mx'\begin{bmatrix}\alpha_1\bar{Q}_1\left(p\right) & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \alpha_n\bar{Q}_n\left(p\right)\end{bmatrix}x\right\}\\
&= m^{-1}\left\{\left(\sum_i x_i\bar{Q}_i\right)^2 - m\sum_i x_i^2\alpha_i\bar{Q}_i\right\}\\
&= m^{-1}\left(\sum_i x_i\bar{Q}_i\right)^2 - m^{-1}\sum_i x_i^2\bar{Q}_i^2 + m^{-1}\sum_i x_i^2\bar{Q}_i^2 - \sum_i x_i^2\alpha_i\bar{Q}_i\\
&= -m^{-1}\underbrace{\left\{\sum_i x_i^2\bar{Q}_i^2 - \left(\sum_i x_i\bar{Q}_i\right)^2\right\}}_{>0} - m^{-1}\left\{\sum_i x_i^2\bar{Q}_i\underbrace{\left(m\alpha_i - \bar{Q}_i\right)}_{>0}\right\} < 0
\end{aligned}
$$

Thus, $\frac{\partial Q(\mathbf{p})}{\partial \mathbf{p}'}$ is negative definite. Therefore, $B$ is negative definite, so that $\psi'\left(\mu^f\right) < 0$.

**Proof of Proposition 2 and 3:**

In this part, I show a reconstruction of demand functions and prove the uniqueness of the solution of the first-order condition. The notations in the following are for Proposition 3 ($1 > a_t > 0$ and $b \geq 0$), but the same proof can be applied for Proposition 2 by setting $b_t = 0$. When $a_t = 0$, the proof is analogous to Theorem 1.

Suppose that we have $1 > a_t > 0$ and $b \geq 0$. Then, we can reconstruct $\bar{h}_{j,t}\left(p_j\right) = v_{j,t}\left(p_j + \frac{b_t}{a_t}\right)^{1-\frac{1}{a_t}}$ for some $v_{j,t}$. Subsequently, the demand function is reconstructed as

$$
\bar{Q}_{j,t}\left(p\right) = m_t\frac{\left(\frac{1}{a_t} - 1\right)v_{j,t}\left(p_j + \frac{b_t}{a_t}\right)^{-\frac{1}{a_t}}}{\sum_{k\in\mathcal{J}} v_{k,t}\left(p_k + \frac{b_t}{a_t}\right)^{1-\frac{1}{a_t}}}.
$$

Note that there exist $v_{j,t}$'s satisfying

$$
\bar{q}_{j,t} = m_t\frac{\left(\frac{1}{a_t} - 1\right)v_{j,t}\left(\bar{p}_j + \frac{b_t}{a_t}\right)^{-\frac{1}{a_t}}}{\sum_{k\in\mathcal{J}} v_{k,t}\left(\bar{p}_k + \frac{b_t}{a_t}\right)^{1-\frac{1}{a_t}}}
$$

for any $j \in \mathcal{J}$. It remains to prove the uniqueness of the solution to the first-order conditions. In the following, I omit time-index, $t$, for simplicity. Now, similar to Theorem 1, by the first-order condition, we have the following for any $j \in J_f$

$$\left\{p_j - \bar{C}_j'\left(\bar{Q}_j\left(\mathbf{p}\right)\right)\right\}\frac{1}{ap_j+b} = \underbrace{1 + m^{-1}\sum_{k\in J_f}\left\{p_k - \bar{C}_k'\left(\bar{Q}_k\left(\mathbf{p}\right)\right)\right\}\bar{Q}_k\left(\mathbf{p}\right)}_{\mu^f} \tag{17}$$

The RHS is rewritten as

$$1 + m^{-1}\sum_{k\in J_f}\underbrace{\left\{p_k - \bar{C}_k'\left(\bar{Q}_k\left(\mathbf{p}\right)\right)\right\}\frac{1}{ap_j+b}}_{\mu^f} am\frac{\left(\frac{1}{a}-1\right)v_k\left(p_k+\frac{b}{a}\right)^{1-\frac{1}{a}}}{\sum_{l\in\mathcal{J}} v_{l,t}\left(p_l+\frac{b}{a}\right)^{1-\frac{1}{a}}}$$

Then, eq. (17) is decomposed into the following system of equations:

$$\left\{p_j - \bar{C}_j'\left(m_t\frac{\left(\frac{1}{a}-1\right)v_j\left(p_j+\frac{b}{a}\right)^{-\frac{1}{a}}}{H_0+H_f}\right)\right\}\frac{1}{ap_j+b} - \mu^f = 0 \ \ \forall j\in\mathcal{J}_f \tag{18}$$

$$\mu^f = 1 + \mu^f\left(1-a\right)\frac{H_f}{H_0+H_f} \tag{19}$$

$$H_f = \sum_{k\in J_f} v_k\left(p_k+\frac{b}{a}\right)^{1-\frac{1}{a}} \tag{20}$$

where $H_0 = \sum_{k\notin J_f}\bar{h}_k\left(p_k\right)$. By eq. (19),

$$\begin{aligned}
&\mu^f = 1 + \mu^f\left(1-a\right)\frac{H_f}{H_0+H_f}\\
\Leftrightarrow\ &\mu^f = \frac{1}{1-\left(1-a\right)\frac{H_f}{H_0+H_f}} = \frac{H_0+H_f}{H_0+H_f-\left(1-a\right)H_f} = \frac{H_0+H_f}{H_0+aH_f} \equiv \mu^f\left(H_f\right)
\end{aligned}$$

Then, by (18),

$$\nu_j\left(p_j,\ H_j\right) \equiv \left\{p_j - \bar{C}_j'\left(m_t\frac{\left(\frac{1}{a}-1\right)v_j\left(p_j+\frac{b}{a}\right)^{-\frac{1}{a}}}{H_0+H_f}\right)\right\}\frac{1}{ap_j+b} - \frac{H_0+H_f}{H_0+aH_f} = 0 \ \ \forall j\in\mathcal{J}_f$$

Note that

$$\frac{\partial \mu^f (H_f)}{\partial H_f} = \frac{1}{H_0 + aH_f} - \frac{H_0 + H_f}{(H_0 + aH_f)^2} a$$
$$= \frac{H_0 + aH_f - a (H_0 + H_f)}{(H_0 + aH_f)^2}$$
$$= \frac{(1 - a) H_0}{(H_0 + aH_f)^2} > 0$$

Note that $p_j$ is uniquely determined by eq. (18) given $H_f$. The solution is denoted as $p_j (H_f)$. Then, the aggregator in eq. (20) is written as

$$H_f = \sum_{k \in J_f} v_{k,\,t} \left( p_k (H_f) + \frac{b_t}{a_t} \right)^{1 - \frac{1}{a_t}} \tag{21}$$

Now, we prove that eq. (21) has a unique solution. It is sufficient to show that the derivative of the RHS of (21) is always less than 1, i.e.,

$$\frac{\partial RHS}{\partial H_f} = \left\{ 1 - \frac{1}{a_t} \right\} \sum_{k \in J_f} v_{k,\,t} \left( p_k (H_f) + \frac{b_t}{a_t} \right)^{-\frac{1}{a_t}} \left[ - \frac{\partial \nu_j (p_j,\, H_f)}{\partial H_f} / \frac{\partial \nu_j (p_j,\, H_f)}{\partial p_j} \right] < 1$$

Observe that

$$\frac{\partial \nu_j (p_j,\, H_f)}{\partial p_j} = \frac{b}{(ap_j + b)^2} + \frac{a}{(ap_j + b)^2} \bar{C}_j' (Q_j) + \frac{1}{(ap_j + b)^2} E$$

where $E = \bar{C}_j'' \left( m \frac{\left(\frac{1}{a} - 1\right) v_j \left(p_j + \frac{b}{a}\right)^{-\frac{1}{a}}}{H_0 + H_f} \right) m \frac{\left(\frac{1}{a} - 1\right) v_j \left(p_j + \frac{b}{a}\right)^{-\frac{1}{a}}}{H_0 + H_f} > 0$ and

$$\frac{\partial \nu_j (p_j,\, H_f)}{\partial H_f} = \frac{1}{(ap_j + b) (H_0 + H_f)} E - \frac{(1 - a) (H_0)}{(H_0 + aH_f)^2}$$

Then, the RHS of eq. (21) becomes

$$\frac{\partial RHS}{\partial H_f} = \left\{1 - \frac{1}{a}\right\} \sum_{k \in J_f} v_k \left(p_k + \frac{b}{a}\right)^{-\frac{1}{a}} \left[-\frac{\partial \nu_j\left(p_j,\, H_f\right)}{\partial H_f} / \frac{\partial \nu_j\left(p_j,\, H_f\right)}{\partial p_j}\right]$$

$$= \underbrace{\{1-a\}}_{\in(0,1)} \underbrace{\sum_{k \in J_f} \frac{v_k\left(p_k + \frac{b}{a}\right)^{1-\frac{1}{a}}}{H_0 + H_f} \underbrace{\left[\frac{E - \frac{(1-a)H_0\left(H_0+H_f\right)(ap_j+b)}{\left(H_0+aH_f\right)^2}}{E + b + a\bar{C}_j'\left(Q_j\right)}\right]}_{<1}}_{<1}$$

$$< 1$$

Thus, the observed prices of firm $f$'s products are the unique solution of the set of the firm $f$'s first-order conditions, given the other's prices.

**Proof of Proposition 4:**

We need to derive the final condition in this proposition.

By the co-evolving property, we can find a permutation such that $\sigma(t) < \sigma(s)$ implies $\epsilon_{j,t}(p) \leq \epsilon_{j,s}(p)$ for all $j \in J$ and for all $p$. If $\bar{p}_{i,t} \leq \bar{p}_{i,s}$, $\bar{p}_{-i,t} \geq \bar{p}_{-i,s}$ and $\sigma(t) < \sigma(s)$, then $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} = \epsilon_{j,t}\left(\bar{p}_{jt},\, \bar{p}_{-jt}\right) \leq \epsilon_{j,t}\left(\bar{p}_{js},\, \bar{p}_{-jt}\right) \leq \epsilon_{j,t}\left(\bar{p}_{js},\, \bar{p}_{-js}\right) \leq \epsilon_{j,s}\left(\bar{p}_{js},\, \bar{p}_{-js}\right) = \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$. Thus, $\alpha_{j,t} - m_t^{-1}\bar{q}_{j,t} \leq \alpha_{j,s} - m_s^{-1}\bar{q}_{j,s}$ $\square$

Table 1: Summary of results

| | Demand | Cost | Extention | Neccesity | Sufficiency | Falsifiability |
|---|---|---|---|---|---|---|
| Theorem 1 | DC | concave | - | Y | Y | N |
| Proposition 1 | Logit | concave | - | Y | Y | Y |
| Proposition 2 | CES | concave | - | Y | Y | Y |
| Proposition 3 | DC with HARA $h$ | concave | - | Y | Y | Y |
| Proposition 4 | *co-evolving* DC | concave | - | Y | N | Y |
| Remark 1 | — | concave | cost shifter | Y | N | — |
| Remark 2 | Logit | concave | collusive competition | Y | N | Y |
| Corollary 2 | DC | concave | full collusion | Y | Y | N |
| Corollary 3 | Logit | concave | full collusion | Y | Y | Y |