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Abstract

We find a necessary and sufficient condition for global stability of the voluntary contribution mechanism with heterogeneous quasilinear preferences under a simultaneous system of difference equations. The condition is that there must exist an outstanding player whose willingness to contribute dominates the sum of the others' willingness.

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1. Introduction

After the Great East Japan Earthquake in 2011, Hamamatsu, a city in the central region of Japan that did not suffer from the earthquake, but expects a large one within a few decades, started building anti-tsunami embankments on more than 17 km based on voluntary public contributions. One company, Ichijo Housing Co., donated JPY 30 billion, and households and other companies, more than 3,000 in total, donated JPY 1 billion. To date, the total donations will cover the cost of the project.

Oxfam reported that eight super-rich men hold the same amount of wealth as the poorest half of the world's population in January 2017[1], and five out of eight are members of the Giving Pledge where they committed to contribute a majority of their wealth to philanthropic causes.[2]

These are examples of the private provision of a public good, with a sizable literature body developed over the past few decades. The basic framework is that each player chooses between consumption of a private good and contribution to it, which is called the *voluntary contribution mechanism* (VCM). In this respect, researchers such as Bergstrom, Blume, and Varian (1986), Bernheim (1986), Cornes and Sandler (1996), and Kotchen (2006) analyze the nature of Nash equilibria, such as the existence, uniqueness, and neutrality. However, the stability of the VCM has not been investigated extensively.

If utility functions are linear, then each player has a dominant strategy with no contribution, making the system stable. However, if they are nonlinear and all players have the same quasilinear utility function and endowment, the system is not asymptotically stable under simultaneous difference equations, and is structurally unstable under simultaneous differential equations as Saijo (2014) shows. That is, the system of the VCM is intrinsically unstable and has a free-riding issue as well.

Since assuming that every player has the same utility function and the endowment is a stringent condition, this paper considers them different. As such, we use quasilinear utility functions that are linear with respect to player *i*'s private good consumption x_i and nonlinear with respect to a public good *y*, that is, $u_i(x_i, y) = x_i + t_i(y)$. The rationale behind this formulation is that the private good represents money and, hence, its marginal utility is constant, but the marginal utility of the public good decreases so that the $t_i(y)$ part is nonlinear. Furthermore, players may have different endowments.

Experimental tests such as Chen and Plott (1996), Chen and Tang (1998), Chen and Gazzale (2004) and Healy (2006) found that players basically use myopic learning dynamics such as best-responses to a recent history of actions. Following them, we will use best response

[1] See https://www.oxfamamerica.org/explore/research-publications/an-economy-for-the-99-percent/

[2] See https://givingpledge.org/#enter

dynamics as the simplest dynamics. If a utility function is quasilinear, the best response function is linear and its slope is -1. The difference in players can be identified by the intercept when no other players contribute, a_i, and let the player who has the maximum value of a_i be player 1. Subsequently, the unique Nash equilibrium is that player 1 contributes a_1 and every other player contributes nothing. Furthermore, a necessary and sufficient condition for *global stability* of the simultaneous system of difference equations is $a_1 > \sum_{j=2}^{n} a_j$, as long as $w_i \geq a_i$, where w_i is the endowment of player *i*. That is, the system is globally stable if there is an outstanding player who can contribute significantly compared with other players[3]. It appears that the anti-tsunami embankment case of Hamamatsu city and the pledgers of the Giving Pledge correspond to this case.

Even if $a_1 \leq \sum_{j=2}^{n} a_j$, the system is locally stable around the Nash equilibrium, but if the number of players increases, the area of initial points that converge to the Nash equilibrium decreases rapidly. That is, the system is *intrinsically unstable* as long as either $a_1 \leq \sum_{j=2}^{n} a_j$ is satisfied, namely players are all alike, or the number of players is large.

The remainder of this paper is organized as follows. Section 2 provides a necessary and sufficient condition for global stability of the VCM, and section 3 analyzes instability and the number of players using an example. Section 4 studies the Cobb-Douglas utility functions, and section 5 describes further research scope.

2. Global stability of the VCM

Let *x* be a private good and *y* be a public good. The production function of the public good is $y = f(x) = x$. That is, for example, one hour of labor input produces one millimeter of anti-tsunami embankment. Player *i* has endowment w_i and must decide to divide w_i into *i*'s own consumption of the private good x_i and contribution s_i to the public good. That is, $y = \sum_1^n s_j$ where $n \geq 2$. This system is called the VCM. We assume that each player has a quasilinear utility function $u_i(x_i, y) = x_i + t_i(y)$. Consequently, player *i* faces the following problem.

$Max\ u_i(x_i, s_i + s_{-i})$ subject to $w_i = x_i + s_i$,

where $s_{-i} = \sum_{j \neq i} s_j$. Let $u_i(w_i - s_i, s_i + s_{-i}) = v_i(s_i, s_{-i}) = w_i - s_i + t_i(s_i + s_{-i})$. A list of contributions $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n)$ is a *Nash equilibrium* if for all *i* $v_i(\hat{s}_i, \hat{s}_{-i}) \geq v_i(s_i, \hat{s}_{-i})$ for all $s_i \in [0, w_i]$. The best response function is defined as

[3] Olson (1965) pointed out that "(i)n smaller groups marked by considerable degrees of inequality - that is, in groups of members of unequal "size" or extent of interest in the collective good - there is the greatest likelihood that a collective good will be provided" (p.34) from the viewpoint of collective good provision rather than from the stability viewpoint.

$r_i(s_{-i}) = \arg\max_{s_i}\{v_i(s_i, s_{-i}) \mid s_i \in [0, w_i]\}$.[4]

As such, the best response functions are linear. In order to show this property, consider the first order condition for the maximization problem, i.e., $\partial v_i / \partial s_i = -1 + \partial t_i / \partial y = 0$. Then, totally differentiating both sides of the condition, we have the slope of the best response function since $\partial(\partial t_i/\partial y)/\partial s_i = \partial(\partial t_i/\partial y)/\partial s_{-i}$:

$$\frac{dr_i}{ds_{-i}} = \frac{ds_i}{ds_{-i}} = -\frac{\partial(\partial t_i/\partial y)/\partial s_{-i}}{\partial(\partial t_i/\partial y)/\partial s_i} = -1.$$

That is, the best response function is linear with -1 slope. Since player *i* cannot contribute a negative value, $r_i(s_{-i}) = \max\{-s_{-i} + a_i, 0\}$, where a_i is the intercept. For simplicity, assume that $a_1 > a_j \geq 0$ for all $j \neq 1$, and $w_i \geq a_i$ for all *i*. Then, since $r_1(0) = a_1$ and $r_j(a_1) = 0$ for all $j \neq 1$, $(a_1, 0, \cdots, 0)$ is a Nash equilibrium. According to Bergstrom, Blume and Varian (1986, 1992), who prove the uniqueness of Nash equilibrium in a general setting, and Bergstrom, Blume and Varian (1986), who show the Nash equilibrium for quasilinear utility functions, we have the following proposition.

Proposition 1. *Suppose that* $a_1 > a_j \geq 0$ *for all* $j \neq 1$ *and* $w_i \geq a_i$ *for all i. Then, the unique Nash equilibrium is* $(a_1, 0, \cdots, 0)$.

The Nash equilibrium is not Pareto efficient. Let $u_i(x_i, y) = x_i + a_i \ln y$.[5] Then, the Samuelson condition is $\sum a_j = y$. Since the public good level at the Nash equilibrium is a_1, it is apparently lower than the Pareto efficient level.

At time *t*, let player *i*'s choice of contribution be s_i^t. We simply assume that player *i* chooses $r_i(s_{-i}^t)$ at time *t* + 1, where *t* = 1, 2, … That is, we assume that every player chooses the best response to the sum of strategies chosen by the other players in the immediately preceding period. Therefore, the stability of the following system must be analyzed.

(1) $s_i^{t+1} = \max\{-s_{-i}^t + a_i, 0\}$ i = 1, …, n and t=1, 2, ….

Ignoring the maximization part, we can rewrite the system in the following manner.

$$\mathbf{s}^{t+1} = \mathbf{A}\mathbf{s}^t + \mathbf{a},$$

[4] If u_i is strictly quasi-concave, the maximizer is unique and r_i is continuous as per Berge's maximum theorem.

[5] The best response function for this utility function is $r_i(s_{-i}) = \max\{-s_{-i} + a_i, 0\}$.

where $\mathbf{A}=\begin{bmatrix} 0 & -1 & \dots & -1 \\ -1 & 0 & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \dots & -1 & 0 \end{bmatrix}$, $\mathbf{s}^t=\begin{bmatrix} s_1^t \\ \vdots \\ s_n^t \end{bmatrix}$ and $\mathbf{a}=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

The stability property is governed by the eigenvalues of **A,** that is, $(1-n,1,\cdots,1)$.[6] Furthermore, as Proposition 1 shows, the Nash equilibrium is *not* located on the linear part of the system and hence, it is *not* a solution of $\mathbf{s}=[\mathbf{I}-\mathbf{A}]^{-1}\mathbf{a}$, where **I** is the identity matrix.[7] That is, understanding the stability property of (1) is a new challenge.

In order to understand the stability of the system, let us introduce the *maximum response function* $m_{II}(s_1)$ of player II, treating all players other than player 1 as one player and denoting the player as player II:

$$m_{II}(s_1)=\begin{cases} (n-1)s_1+\sum_{j\neq 1}^{n} a_j \text{ for } 0\le s_1<a_n \\ (n-2)s_1+\sum_{j\neq 1}^{n-1} a_j \text{ for } a_n\le s_1<a_{n-1} \\ \cdots \\ -2s_1+a_2+a_3 \text{ for } a_4\le s_1<a_3 \\ -s_1+a_2 \text{ for } a_3\le s_1<a_2 \\ 0 \text{ for } a_2\le s_1 \end{cases},$$

which is a continuous piecewise linear function. This function is *not* the best response function of all players other than player 1, but shows the maximum possible sum of contributions given s_1.

Consider the following example. Let $(a_1,a_2,a_3)=(10,6,2)$ and $w_i=12$ for all *i*. Then, m_{II} is a function summing up $\max\{-s_1+6,0\}$ and $\max\{-s_1+2,0\}$ vertically. This is *a-b-c-e* in Figure 1, where the vertical axis, s_{II}, represents the range of $m_{II}(s_1)$ for player II and s_{-1} for player 1. The slope of *a-b* is -2, and the slope of *b-c* is -1. If s_1 is greater than 6, neither players 2 nor 3 contribute, and, hence, the *c-e* part is flat.

Let $s^1=(s_1^1,s_2^1,s_3^1)=(1,5,0)$, which corresponds to s^1 in Figure 1. Then, the best response is (5,5,0) which corresponds to *q*. Consider another initial point such as $\tilde{s}^1=(1,3,2)$, which also corresponds to s^1. The best response is (5,3,0), which corresponds to *p*. Let $\hat{s}^1=(1,1,4)$, which still corresponds to s^1. Then, the best response is (5,1,0), which corresponds to *k*. Since $s_2^2=\max\{-(s_1^1+s_3^1)+6,0\}$ and $s_3^2=\max\{-(s_1^1+s_2^1)+2,0\}$, for $s_1^1=1$ the maximum possible response s_2^2 is 5, and the maximum possible response s_3^2 is 1. The sum of 5 and 1 shows $m_{II}(1)$ or *d*. That is, the range of reactions of players 2 and 3 for $s_1=1$ is from 0 to $m_{II}(1)$, which is *d-f*.

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[6] See Saijo, Feng and Kobayashi (2017).

[7] Note that every element of **I - A** is 1 and, hence **I-A,** is not invertible.

Given $s^1 = (s^1_1, s^1_{-1})$, the set of best responses $\{(a_1 - s^1_{-1}, b) : b \in [0, m_{II}(s^1_1)]\}$ to s^1 is denoted by $M(s^1)$, and called *the set of maximum possible best responses to* s^1. Since the reaction of player 1 to s^1 is 5, $M(s^1)$ should be *g-h*. Three best responses, such as *q*, *p*, and *k*, are in $M(s^1)$. If $s^1 = d$, then $M(d)$ is *d′-f′*, and if $s^1 = f$, then $M(f)$ is *d″-e*. That is, if s^1 is somewhere on *d-f*, then the set of the maximum possible best responses to *d-f* is square *d′-f′-e-d″*.

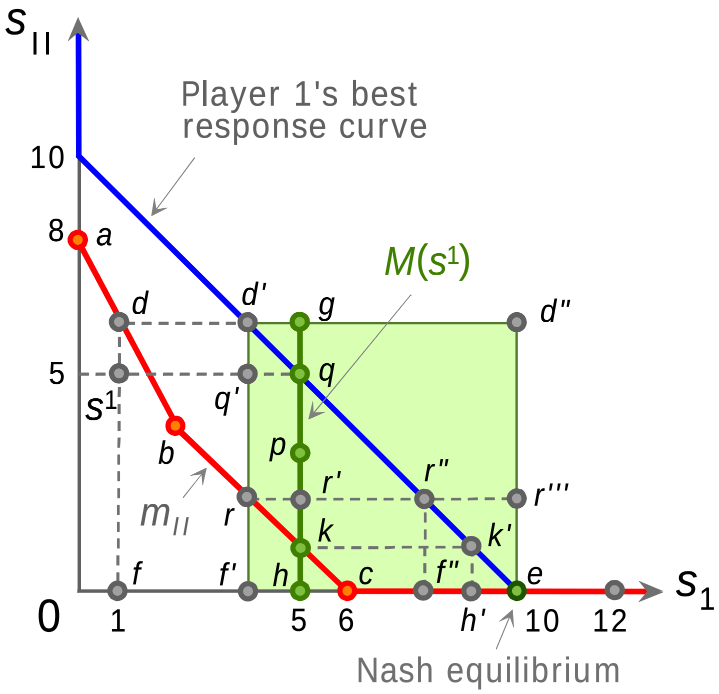

Figure 1. Best response areas.

Consider now what is the next set of maximum possible responses to square *d′-f′-e-d″* in Figure 1. Point *d′* becomes line *r-f′* and *q′* becomes *r′-h*. Furthermore, *r* becomes *r″-f″*, and any point between *r* and *f′* becomes a line parallel to *r″-f″* and located between *f″-e*. That is, the maximum possible best responses to line *d′-f′* is rectangle *r-f′-e-r‴*. Now, consider a point in *d′-r-r‴-d″*, such as *q*. Point *q* becomes line *k-h*, and this line is included in rectangle *r-f′-e-r‴* since *r-c-e* is included in *r-f′-e-r‴*. Consider any point such as *k* in *r-f′-e-r‴*. Point *k* becomes line *k′-h′* in *r-f′-e-r‴*. That is, the set of maximum possible responses to square *d′-f′-e-d″* is rectangle *r-f′-e-r‴*. It is now easy to see that the set of maximum possible responses to rectangle *r-f′-e-r‴* becomes square *r″-f″-e-r‴*. That is, if this sequence of sets converges to the Nash equilibrium, the original sequence must converge to it. The strategy for showing the stability of the system is described as follows.

First, consider Figure 2 and take any s^1. Then, s^2 must be in rectangle *a-0-e-b* in Figure 2, since the largest maximum possible best response set to s_1^1 is $[0, \sum_{j=2}^{n} a_j]$ and the largest maximum possible best response set to $\sum_{j=2}^{n} s_j^1$ is $[0, a_1]$. Consider now the set of maximum possible best responses to *a-0-e-b.* The maximum possible best responses to line *a*-0 covers square *c-f-e-b*, the maximum possible best responses to *a-0-f-c* are in *c-f-e-b*, and the maximum possible best responses to *c-f-e-b* are in *c-f-e-b.* Consider the next set of maximum possible best responses to square *c-f-e-b.* The maximum possible best responses to line *c-f* are in rectangle *d-f-e-p*, the maximum possible best responses to *c-d-p-b* are in rectangle *d-f-e-p*, and the maximum possible best responses to *d-f-p-e* are in *d-f-p-e*. Similarly, the next set of maximum possible best responses to rectangle *d-f-p-e* is square *g-h-e-p*. Repeating the same procedure, *g-h-e-p* becomes rectangle *k-h-e-q*, and *k-h-e-q* becomes square *r-w-e-q*. Since the base of square *r-w-e-q* is on the flat part of m_{II}, the next rectangle is line *w-e*. Then, any best response point on *w-e* becomes one-point square *e*, which is the Nash equilibrium. Stability here is *global stability*: any initial point converges to some equilibrium point.[8]

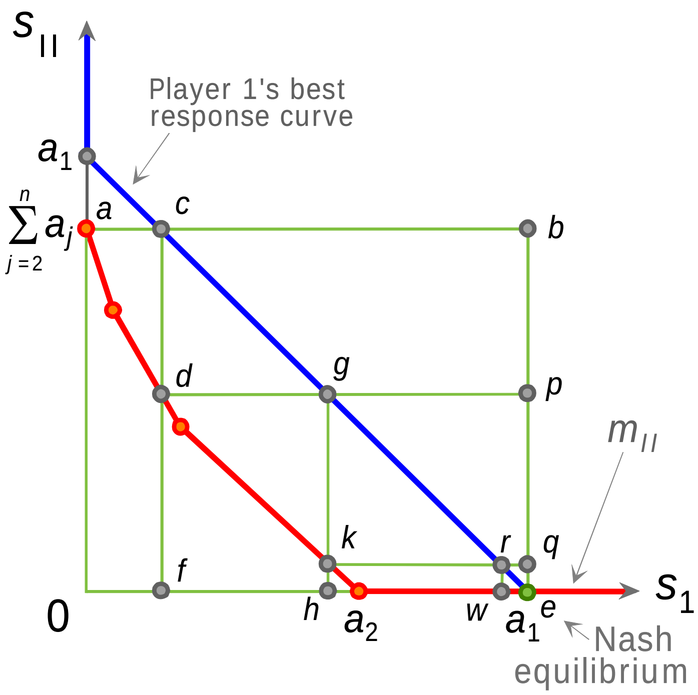

Figure 2. From square to rectangle.

Let us introduce two concepts. $Rec(s_1)$ is the minimum rectangle containing two points $(s_1, m_{II}(s_1))$ and $(a_1, 0)$, where a rectangle is a set containing the interior and the boundary, and

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[8] See Arrow and Hurwicz (1958) and Arrow, Block, and Hurwicz (1959).

$sq(s_1)$ is the minimum square containing two points $(s_1, a_1 - s_1)$ and $(a_1, 0)$, where a square is a set containing the interior and the boundary. As such, we have the following lemma.

Lemma 1. *Suppose that $w_i > a_i$ for all i and $a_1 > \sum_{j=2}^{n} a_j$. Let $s_1 \in [0, a_1)$. Then,*
(i) the set of maximum possible best responses to $rec(s_1)$ is $sq(a_1 - m_{II}(s_1))$ and $sq(a_1 - m_{II}(s_1))$ is a proper subset of $rec(s_1)$; and
(ii) the set of maximum possible best responses to $sq(s_1)$ is $rec(s_1)$ and $rec(s_1)$ is a proper subset of $sq(s_1)$.

Proof. (i) Since $w_i > a_i$ for all i, player i can choose a strategy up to a_i. Take any $s_1 \in [0, a_1)$. Since $a_1 > \sum_{j=2}^{n} a_j$ and the construction of m_{II}, player 1's best response curve is always above m_{II}, except for $s_1 = a_1$, i.e., $m_{II}(s_1) < a_1 - s_1$ and hence $s_1 < a_1 - m_{II}(s_1)$. Since s_1 is the value of the horizontal axis of the left bottom vertex of $rec(s_1)$, $a_1 - m_{II}(s_1)$ is the value of the left bottom vertex of $sq(a_1 - m_{II}(s_1))$. Therefore, the base of $sq(a_1 - m_{II}(s_1))$ is a proper subset of the base of $rec(s_1)$. Since the height of $sq(a_1 - m_{II}(s_1))$ is $m_{II}(s_1)$, because $a_1 - (a_1 - m_{II}(s_1)) = m_{II}(s_1)$, and the height of $rec(s_1)$ is $m_{II}(s_1)$ by definition, both have the same height. That is, $sq(a_1 - m_{II}(s_1))$ is a proper subset of $rec(s_1)$.

In order to show that the set of maximum possible best responses to $rec(s_1)$ is $sq(a_1 - m_{II}(s_1))$, consider three possible areas of $rec(s_1)$. Consider first any point (s_1, b) where $b \in [0, m_{II}(s_1)]$. Then, $M(s_1, b) = \{(a_1 - b, b') : \text{for some } b' \in [0, m_{II}(s_1)]\}$. Since $(a_1 - b) - (a_1 - m_{II}(s_1)) = m_{II}(s_1) - b \geq 0$, $M(s_1, b)$ is in $sq(a_1 - m_{II}(s_1))$, and hence, the set of maximum possible best responses to (s_1, b) with $b \in [0, m_{II}(s_1)]$ is exactly the same as $sq(a_1 - m_{II}(s_1))$.

Consider any point (s_1', b) where $s_1' \in [s_1, a_1 - m_{II}(s_1)]$ and $b \in [0, m_{II}(s_1)]$. Since m_{II} is a non-increasing function, $m(s_1') \leq m(s_1)$. Since $a_1 - m_{II}(s_1) \leq a_1 - m_{II}(s_1')$, $M(s_1', b)$ must be in $sq(a_1 - m_{II}(s_1))$.

Finally, consider any point (s_1', b) where $s_1' \in [a_1 - m_{II}(s_1), a_1]$ and $b \in [0, m_{II}(s_1)]$. Applying the same argument above, $M(s_1', b)$ must be in $sq(a_1 - m_{II}(s_1))$.
(ii) Take any s_1. Since $a_1 > \sum_{j=2}^{n} a_j$, the construction of m_{II}, and player 1's best response curve is always above m_{II} except for $s_1 = a_1$, $rec(s_1)$ is a proper subset of $sq(s_1)$. Since m_{II} is a non-increasing function, $m(s_1') \leq m(s_1)$ for all $s_1' \in [s_1, a_1]$, and hence, $M(s_1', b)$ with $s_1' \in [s_1, a_1]$ and $b \in [0, a_1 - s_1]$ must be in $rec(s_1)$.■

Proposition 2 shows a necessary and sufficient condition for stability.

Proposition 2. *Suppose that* $a_1 > a_j \geq 0$ *for all j* $\neq$ *1 and* $w_i \geq a_i$ *for all i. Then, the simultaneous system of difference equations is globally stable if and only if* $a_1 > \sum_{j=2}^{n} a_j$.

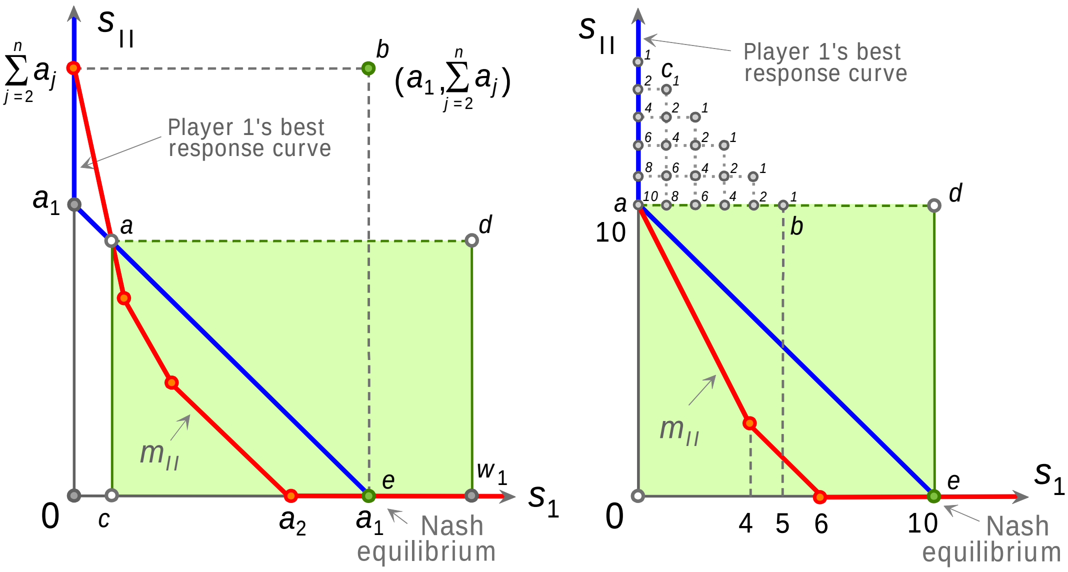
Proof. (i) The "if" part. Take any initial point s^1. Then, the best response s^2 to s^1 must be in *rec*(0). By lemma 1, the next set of maximum possible best responses to *rec*(0) is $sq\,(a_1 - \sum_{j=2}^{n} a_j)$, and hence, the height of the square is $\sum_{j=2}^{n} a_j$. Let $c = a_1 - \sum_{j=2}^{n} a_j > 0$. Since the slope of m_{II} is at most -1 as far as $s_1 \in [0, a_2]$, the sequence from rec(0) to $sq\,(a_1 - \sum_{j=2}^{n} a_j)$ and the square to the next rectangle ends in a finite step, due to the fact that the height of a square shrinks by at least *c* to the next square. Once the base of a square reaches an interval $[a_2, a_1]$ on the s_1 axis that is not a point since $a_1 > a_2$, the next rectangle is an interval contained in $[a_2, a_1]$. As such, the best response to the interval must be the Nash equilibrium.

(ii) The "only if" part. It suffices to show that if $a_1 \leq \sum_{j=2}^{n} a_j$, the system is unstable. Let $s^1 = (a_1, a_2, ..., a_n)$. Then, since $a_1 \leq \sum_{j=2}^{n} a_j$, the best response is (0, 0, …, 0). The next best response is $(a_1, a_2, ..., a_n)$ and hence, the system is not stable. ∎

3. Instability and the number of players: an example

Notice that stability in Proposition 2 is not local, but *global*. That is, as far as $a_1 < \sum_{j=2}^{n} a_j$ is satisfied, the system goes to the Nash equilibrium in a few steps starting from *any* initial point. When $a_1 \geq \sum_{j=2}^{n} a_j$, the system is not globally stable. However, a sequence starting from an initial point in a rather large area such as *a-c-w*1*-d* in Figure 3-(a) converges to the Nash equilibrium. In this sense, the system is *locally* stable.

The stability on the boundaries of *a-c-w*1*-d* in Figure 3-(*a*) is rather complex, since we translate the *n* dimensional space into the two dimensional space. Let $(a_1, a_2, a_3) = (10, 6, 4)$ and $w_i = 10$ for all *i*. Then, the Nash equilibrium is (10, 0, 0), as shown in Figure 3-(*b*), but the system is not stable since $a_1 \leq a_2 + a_3$. Assume that each player can announce only integers. Consider $s^1 = (0, 4, 6)$, that is *a* in the figure. Then, $s^2 = (0, 0, 0)$, $s^3 = (10, 6, 4)$ and $s^4 = (0, 0, 0)$. That is, this sequence alternates between 0 and *d*, and hence, it does not converge to the Nash equilibrium. On the other hand, consider $s^1 = (0, 3, 7)$ that is also *a* in the figure. As such, $s^2 = (0, 0, 1)$, $s^3 = (9, 5, 4)$, $s^4 = (1, 0, 0)$, $s^5 = (10, 5, 3)$, $s^6 = (2, 0, 0)$, $s^7 = (10, 4, 2)$, $s^8 = (4, 0, 0)$, $s^9 = (10, 2, 0)$, $s^{10} = (8, 0, 0)$, and $s^{11} = (10, 0, 0)$. That is, this sequence converges to the Nash equilibrium. The upper right number of *a* is the number of initial points that converge to the Nash equilibrium. Consider $s^1 = (5, 9, 1)$, that is *b* in the figure. Then, $s^2 = (0, 0, 0)$, $s^3 = (10, 6, 4)$, and $s^4 = (0, 0, 0)$. That is, this sequence does not converge to the Nash equilibrium. On the other hand, consider $s^1 = (5, 10, 0)$, that is also *b* in the figure. Consequently, $s^2 = (0, 1, 0)$, $s^3 = (9, 6, 3)$, and $s^4 = (1, 0, 0)$. For the rest of the sequence, see the sequence starting from $s^1 = (0, 3, 7)$, which converges to the Nash equilibrium.



(a) (b)

Figure 3. The stable initial point area when the system is not stable.

Consider points that are not in $\overline{a\text{-}0\text{-}e\text{-}d}$, where the bar indicates the closure of *a*-0-*e*-*d*. If each element of s^1 is rather large, such as $s^1 = (7,8,9)$, then s^2 is at 0, i.e., $s^2 = (0,0,0)$. Therefore, the sequence repeats *d* and 0. However, there is a chance for some s^2 to be in the stable part of *a-0-e-d* even though s^1 is not in $\overline{a\text{-}0\text{-}e\text{-}d}$. For example, consider $s^1 = (1,10,4)$, that is *c* in the figure. Then, $s^2 = (0,1,0)$ and the rest are the same in the above paragraph. That is, the sequence converges to the Nash equilibrium.

Consider the number of initial points satisfying $0 \le s_1^1 \le 10$ and $0 \le s_2^1 + s_3^1 \le 10$, that is, the number of all possible initial points of $\overline{a\text{-}0\text{-}e\text{-}d}$. As such, since the number of initial points satisfying $0 \le s_2^1 + s_3^1 \le 10$ is 66,[9] and the number of possible choices of player 1 is 11, the number of all possible initial points of $\overline{a\text{-}0\text{-}e\text{-}d}$ is $726 = 66 \times 11$. On the other hand, the number excluding $\overline{a\text{-}d}$ in $\overline{a\text{-}0\text{-}e\text{-}d}$ is $605 = 55 \times 11$. Furthermore, the number of initial points that converge to the Nash equilibrium is 680, and the number of initial points that are not in $\overline{a\text{-}0\text{-}e\text{-}d}$ and converge to the Nash equilibrium is 45 (the sum of small numbers where the vertical axis value is at least 11 in Figure 3-(b)). That is, the number of initial points that are in $\overline{a\text{-}0\text{-}e\text{-}d}$ and converge to the Nash equilibrium is 635. Furthermore, since the number of initial points on *a*-*d* that converge to the Nash equilibrium is 31, 605 + 31 - 1 = 635, that is consistent with the previous number where "1" is for the origin that does not converge to the Nash equilibrium. Generally, initial points that are in *a*-0-*e*-*d* or that are "close" to *a*-0-*e*-*d* converge to the Nash equilibrium.

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[9] The computation is done in *Mathematica* 9.

Although the system is not stable due to the distribution of a_i, if the number of initial points in a-c-w_1-d in Figure 3-(a) is relatively large compared with the total number of all possible initial points, instability is not a significant issue. Considering the previous example and assuming that each player can announce only integers, let the number of initial points in $\overline{a\text{-}0\text{-}e\text{-}d}$ be a surrogate number that converges to the Nash equilibrium. Table 1 shows the number of players n, the number of initial points A_n satisfying $0 \le s_1^1 \le 10$ and $0 \le \sum_{j\ne1}^{n} s_j^1 \le 10$, A_{n+1}/A_n, and $A_n/11^n$. Note that 11^n are the all possible initial points when the number of players is n. Then, $A_n/11^n$ is a rough ratio of initial points that converges to the Nash equilibrium. As the number of players increases, the ratio rapidly decreases. When n is 5, 6, 7 or 8, the ratio is 7%, 2%, 0.5% or 0.1%, respectively. This is because the number of all possible initial points increases eleven times as the number of players increases, while the number of initial points in $\overline{a\text{-}0\text{-}e\text{-}d}$ increases less as the number of players increases (see $\{A_{n+1}/A_n\}$ in Table 1). That is, instability of the VCM is a serious problem even the number of players is below 8.

Table 1. Possible stable ratio when the system is not stable.

The number of players (n)	The number of initial points (A_n) satisfying $0 \le s_1^1 \le 10$ and $0 \le \sum_{j\ne1}^{n} s_j^1 \le 10$	A_{n+1}/A_n	$A_n/11^n$
3	726		$726/11^3 = 0.5456$
4	3,146	3,146/726=13/3=4.33	$3{,}146/11^4 = 0.2149$
5	11,011	11,011/3,146=3.5	$3{,}146/11^5 = 0.0684$
6	33,033	33,033/11,011=3.0	$33{,}033/11^6 = 0.0186$
7	88,088	88,088/33,033=8/3=2.67	$88{,}088/11^7 = 0.0045$
8	213,928	213,928/88,088=17/7=2.43	$213{,}928/11^8 = 0.0010$

4. The Cobb-Douglas case: Examples

Several experimentalists, such as Andreoni (1993), Chan, Mestelman, Moir, and Muller (1996), Cason, Saijo, and Yamato (2002), and Sutter and Weck-Hannemann (2004), use Cobb-Douglas utility functions assuming that every players has the same utility function. As such, it could be useful to summarize the stability property when players have heterogeneous Cobb-Douglas utility functions.

Let player i's utility function be $v_i(s_i, s_{-i}) = (w - s_i)^{\alpha_i}(s_i + s_{-i})^{1-\alpha_i}$, where $0 < \alpha_i < 1$. Then, the first order condition is $\partial v_i / \partial s_i = -\alpha_i((s_i + s_{-i})/(w - s_i))^{1-\alpha_i} + (1-\alpha_i)((w - s_i)/(s_i + s_{-i}))^{\alpha_i} = 0$. That is, $(s_i + s_{-i})/(w - s_i) = (1-\alpha_i)/\alpha_i$, and, hence, $s_i = -\alpha_i s_{-i} + (1-\alpha_i) w_i$. Therefore the system is
$\mathbf{s}^{t+1} = \mathbf{A}\mathbf{s}^t + \mathbf{a}$,

where $\mathbf{A} = \begin{bmatrix} 0 & -\alpha_1 & \cdots & -\alpha_1 \\ -\alpha_2 & 0 & & \vdots \\ \vdots & & \ddots & -\alpha_{n-1} \\ -\alpha_n & \cdots & -\alpha_n & 0 \end{bmatrix}$, $\mathbf{s}^t = \begin{bmatrix} s_1^t \\ \vdots \\ s_n^t \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} (1-\alpha_1)w_1 \\ (1-\alpha_2)w_2 \\ \vdots \\ (1-\alpha_n)w_n \end{bmatrix}$.

The asymptotic stability of the system is governed by the eigenvalues of **A,** assuming that the Nash equilibrium is in the interior. For example, let $(\alpha_1, \alpha_2, \alpha_3) = (3/4, 1/2, 2/5)$ and $(w_1, w_2, w_3) = (20, 9, 8)$. Then, $\det[\mathbf{I}\text{-}\mathbf{A}] \neq 0$, the Nash equilibrium is (7/17, 42/17, 62/17), which is in the interior, and the eigenvalue vector is (1.074, -0.634, -0.441). Hence, the system is not asymptotically stable since some of the absolute values of eigenvalues are above 1. Let $(\alpha_1, \alpha_2, \alpha_3) = (3/5, 1/2, 2/5)$ with the same endowment vector. Then, $\det[\mathbf{I}\text{-}\mathbf{A}] \neq 0$ and the Nash equilibrium is (167/25, 3/25, 52/25), which is in the interior, and the eigenvalue vector is (0.991, -0.554, -0.437). Hence the system is asymptotically stable. That is, the stability of the system is an important issue when utility functions are Cobb-Douglas.

5. Concluding remarks

We find a necessary and sufficient condition for global stability of the VCM with heterogeneous quasilinear preferences under simultaneous difference equations. The implication is that there must exist an eminent player whose willingness to totally contribute dominates the sum of the willingness to contribute of the rest. Since the anti-tsunami embankment case is rare, the VCM is not globally stable. Although the VCM is not globally stable, there are initial points that converge to the Nash equilibrium. However, the size of these points relative to all possible initial points diminishes rapidly as the number of players increases. If we consider that these two points are generic, the VCM perspective is bleak. That is, it is intrinsically unstable, and, hence, its applicability problematic.

The validity of our findings can be tested by experiments. Although there are many experimental results, almost none use heterogeneous and nonlinear payoff functions. Additionally, the number of players used in VCM experiments is at most five to eight, excluding experiments with large number of subjects. Consequently, conducting experimental research in this area is required.

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