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Nash Implementation in Production Economies with Unequal Skills: A Characterization*

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Abstract

The present study examines production economies with unequal labor skills, where the planner is ignorant of the set of feasible allocations in advance of production. In particular, we characterize Nash implementation by canonical mechanisms by means of *Maskin monotonicity* and a new axiom, *non-manipulability of unused skills* (**NUS**), where the latter represents a weak independence property with respect to changes in skills. Following these characterizations, we show that some Maskin monotonic social choice correspondences are not implementable if information about individual skills is absent.

JEL Classification Codes: C72, D51, D78, D82

Keywords: Unequal labor skills; Nash implementation; Canonical mechanisms; Non-manipulability of unused skills

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1 Introduction

In this paper, Nash implementation of desired resource allocations is discussed in production economies with possibly unequal labor skills, which are unknown to the planner (the mechanism coordinator). A typical example of such economies is fisheries, where the mechanism design for Nash implementation is of practical interest. In fisheries, the freedom of operation may lead each individual fisher to overexploit resources with scant regard for the future sustainability of fishing stocks. Therefore, countries that share fishing grounds work on resource management and employ incentive schemes to control individual operations. For instance, in Norway, the harvesting of marine resources is regulated to ensure that source stocks are self-renewable.[1][2][3][4]

Most of the vast literature on implementation theory presumes that while social planners cannot know each individual's preferences, they do know the set of feasible alternatives. In economic environments, however, examples abound of resource allocation problems in which the planner may not know *in advance* the set of feasible alternatives. In this case, each individual's private information consists of not only his or her preferences but also his or her endowments and/or human capital. In such problems, as Jackson (2001) pointed out, the set of feasible alternatives (feasible allocations) is endogenously determined based on individuals' strategies, over which the social planner may have no control. Given this setting, it is necessary to extend

[1]Sustainable management requires knowledge on the size of the stocks, their age composition, their distribution, and the environment in which they live. Every year, data from Norwegian scientific surveys and from fishers are compared with data from other countries (Norwegian marine scientists cooperate closely with researchers from other countries, especially Russia) and assessed by ICES, the International Council for the Exploration of the Sea.

[2]This practice is in accordance with international agreements including the 1982 UN Law of the Sea Convention, the 1995 UN Fish Stocks Agreement, and the 1995 FAO Code of Conduct for Responsible Fisheries.

[3]Recently, an ecosystem approach is increasingly being applied to Norwegian fisheries management. This takes into account not only how harvesting affects fish stocks, but also how fisheries affect the marine environment for living marine resources in general.

[4]Given that the total allowable catch in the Barents Sea is allocated through negotiations under international agreements, the country's quotas are distributed among different groups of fishers and then subdivided and allocated among fishing boats in each group. With this in mind, the *resource management mechanism* in Norway is expected to implement fishing allocations by monitoring and punishing each fishing boat's overexploitation of resources.

the classical framework of implementation theory into a framework with endogenous feasible allocations that allows each individual to misrepresent not only his or her preferences but also his or her endowments or human capital.

Hurwicz et al. (1995) analyzed endogenous feasible allocations in Nash implementation under economic environments, while Tian and Li (1995), Hong (1995), and Tian (1999, 2000, 2009) all addressed the above issue by designing a mechanism to implement a specific *social choice correspondence* (SCC) such as the Walrasian solution. In these works, each individual is allowed to *understate* (or withhold) his or her own material endowments, but he or she is not allowed to overstate them, since the planner is assumed to require individuals to "place the claimed endowments on the table" [Hurwicz et al. (1995)].

One of the essential features of production economies with unknown skills, however, is that each individual is allowed to not only understate but also *overstate* his or her endowment of labor skill. Because in this framework the planner cannot require individuals to place the claimed endowments of their skills on the table in advance of production. Another prominent feature of such economies, as discussed by Yamada and Yoshihara (2008), is that the planner may not be authorized to allocate labor hours among agents consistent with the desired allocations of consumption bundles and leisure. Thus, the main role of the planner would be to simply allocate the outputs produced among agents while leaving to them decisions about their supplies of labor hours in the production process. Among the several studies[5] of implementation in production economies, some authors such as Yamada and Yoshihara (2007, 2008) address this essential feature and discuss implementation by mechanisms with price-quantity messages.

In contrast, this paper studies Nash implementation in production economies with unknown skills, by considering a much broader class of available mechanisms. The restriction that specifies the class of available mechanisms in this paper is very subtle, in that it allows the available mechanisms to contain the class of *canonical mechanisms*. Here, a mechanism is called *canonical* if it makes each agent announce a profile of utility functions. In addition, all available canonical mechanisms are required to be *forthright* [Saijo, Tatamitani, and Yamato (1996); Lombardi and Yoshihara (2013)]. It is well-known that

[5] In addition to the above-mentioned studies, for instance, Suh (1995), Yoshihara (1999), Kaplan and Wettstein (2000), and Tian (2009) all proposed simple or natural mechanisms to implement particular SCCs, whereas Shin and Suh (1997) and Yoshihara (2000) characterized SCCs implementable by simple or natural mechanisms.

Maskin monotonicity [Maskin (1999)] is the necessary and sufficient condition for Nash implementation in production economies when skills are known to the planner. Moreover, for this characterization, constructing a mechanism within the class of canonical ones with forthrightness is sufficient.

However, it is uncertain whether such a property is still preserved in production economies with unknown skills. First, remember that, in the literature of Nash implementation, Maskin monotonicity is usually defined with an exogenously fixed set of feasible allocations. However, as the set of feasible allocations is endogenously determined in production economies with unknown skills, it is unclear how Maskin monotonicity can be defined. Second, in this paper's framework, as economies evolve not only because of changes in utility functions but also owing to changes in skills, Nash implementability of SCCs should be examined when the unknown skills are changed. This new feature may require another axiom to characterize Nash implementation in this context.

This paper provides a reformulation of Maskin monotonicity, called *Monotonicity* (**M**), which is suited to the case of production economies with unknown skills. Moreover, a new axiom called *non-manipulability of unused skills* (**NUS**) is introduced. This axiom stipulates the behavior of SCCs with respect to some specific changes in skills. It has an independence property with respect to skill changes in a quite weak sense, as discussed in section 3 later.

This paper shows that Nash implementability of interior and efficient SCCs by canonical mechanisms is fully characterized by **M** and **NUS**. Note that Nash implementation by canonical mechanisms is equivalent to Nash implementation whenever the production function is strictly concave. This is because **NUS** is shown to be vacuously satisfied in economies with strictly concave production functions. However, the latter axiom is by no means trivial. There is an economically meaningful SCC, called the *maximal workfare solution*, which satisfies **M**, but not **NUS** in economies with linear production functions. Thus, although this solution is implementable in the classical framework where the endowments of skills are known to the planner, it is non-implementable in the extended framework where they are unknown.

The remainder of this paper is organized as follows. Section 2 defines the model. Section 3 fully characterizes Nash implementation by the canonical mechanisms, and Section 4 offers some examples of implementable and non-implementable SCCs. Concluding remarks are presented in Section 5.

2 The Basic Model

2.1 Economic Environments and SCCs

There are two goods, one of which is an input (labor time) $x \in \mathbb{R}_+$ to be used to produce the other good $y \in \mathbb{R}_+$.[6] There is a set $N = \{1, \ldots, n\}$ of agents, where $2 \leq n < +\infty$ holds in general unless a further specification is imposed. Each agent i's consumption is denoted by $z_i = (x_i, y_i)$, where x_i denotes labor time and y_i output. All agents face a common upper bound of labor time $\bar{x}$, where $0 < \bar{x} < +\infty$, and so they have the same consumption set $Z \equiv [0, \bar{x}] \times \mathbb{R}_+$.

Each i's preferences are defined on Z and represented by a utility function $u_i : Z \rightarrow \mathbb{R}$, which is continuous and quasi-concave on Z and strictly monotonic (decreasing in labor time and increasing in the share of output) on $\overset{\circ}{Z} \equiv [0, \bar{x}) \times \mathbb{R}_{++}$.[7] Moreover:

Assumption 1: $\forall i \in N$, $\forall z_i \in \overset{\circ}{Z}$, $\forall z'_i \in Z \backslash \overset{\circ}{Z}$, $u_i(z_i) > u_i(z'_i)$.

We use $\mathcal{U}$ to denote the class of such utility functions.

Each i has **labor skill** $s_i \in \mathbb{R}_{++}$. The universal set of skills for each agent is denoted by $\mathcal{S} = \mathbb{R}_{++}$.[8] Labor skill $s_i \in \mathcal{S}$ is i's **effective labor supply** per hour measured in efficiency units. This skill can also be interpreted as i's **labor intensity** exercised in production.[9] Thus, if the agent's **labor time** is $x_i \in [0, \bar{x}]$ and labor skill $s_i \in \mathcal{S}$, then $s_i x_i \in \mathbb{R}_+$ denotes the agent's **effective labor contribution** to production measured in efficiency units. The production technology is a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is continuous, strictly increasing, and concave such that $f(0) = 0$. For simplicity, we fix

[6] The symbol $\mathbb{R}_+$ denotes the set of non-negative real numbers.

[7] The symbol $\mathbb{R}_{++}$ denotes the set of positive real numbers.

[8] For any two sets X and Y, $X \subseteq Y$ whenever any $x \in X$ also belongs to Y, and $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

[9] It might be more natural to define labor skill and labor intensity in a discriminative way: for example, if $\overline{s}_i \in \mathcal{S}$ is i's labor skill, then i's labor intensity is a variable s_i, where $0 < s_i \leq \overline{s}_i$. In such a formulation, we may view the amount of s_i as being determined endogenously by agent i. In spite of this more natural view, we assume in the following discussion that labor intensity is a constant value, $s_i = \overline{s}_i$, for the sake of analytical simplicity. The main theorems in the following discussion remain valid with a few changes to the settings of the economic environments, even if labor intensity was assumed to be varied.

f. Thus, an **economy** is a pair of profiles $\boldsymbol{e} \equiv (\boldsymbol{u},\boldsymbol{s})$ with $\boldsymbol{u} = (u_i)_{i\in N} \in \mathcal{U}^n$ and $\boldsymbol{s} = (s_i)_{i\in N} \in \mathcal{S}^n$. Denote the class of such economies by $\mathcal{E} \equiv \mathcal{U}^n \times \mathcal{S}^n$.

Given $\boldsymbol{s} \in \mathcal{S}^n$, an allocation $\boldsymbol{z} = (x_i, y_i)_{i\in N} \in Z^n$ is **feasible for $\boldsymbol{s}$** if $\sum y_i \leq f\left(\sum s_i x_i\right)$. Denote by $Z(\boldsymbol{s})$ the set of feasible allocations for $\boldsymbol{s} \in \mathcal{S}^n$. Given $\boldsymbol{s} \in \mathcal{S}^n$, a feasible allocation $\boldsymbol{z} \in Z(\boldsymbol{s})$ is **interior** if $z_i \in \overset{\circ}{Z}$ for all $i \in N$. Denote by $\overset{\circ}{Z}(\boldsymbol{s})$ the set of interior feasible allocations for $\boldsymbol{s} \in \mathcal{S}^n$. An allocation $\boldsymbol{z} = (z_i)_{i\in N} \in Z^n$ is **Pareto-efficient for $\boldsymbol{e}$** $= (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$ if $\boldsymbol{z} \in Z(\boldsymbol{s})$ and there does not exist $\boldsymbol{z}' = (z'_i)_{i\in N} \in Z(\boldsymbol{s})$ such that for all $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, and for some $i \in N$, $u_i(z'_i) > u_i(z_i)$. Let $P(\boldsymbol{e})$ denote the set of Pareto-efficient allocations for $\boldsymbol{e} \in \mathcal{E}$. Let the unit simplex $\Delta \equiv \{p = (p_x, p_y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid p_x + p_y = 1\}$ be the set of price vectors, where p_x represents the price of labor (measured in efficiency units) and p_y the price of output. Then, a price vector $p \in \Delta$ is an **efficiency price for $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{e})$ at $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in E$** if (i) for each $x' \in R_+$, $p_y f(x') - p_x x' \leq \sum (p_y y_i - p_x s_i x_i)$; and (ii) for each $i \in N$ and each $z'_i \in Z$, $u_i(z'_i) \geq u_i(z_i)$ implies $p_y y'_i - p_x s_i x'_i \geq p_y y_i - p_x s_i x_i$. The set of efficiency prices for $\boldsymbol{z}$ at $\boldsymbol{e}$ is denoted by $\Delta^P(\boldsymbol{e}, \boldsymbol{z})$.

A **social choice correspondence** (SCC) or **solution** is a mapping $\varphi : \mathcal{E} \twoheadrightarrow Z^n$ such that for each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, $\varnothing \neq \varphi(\boldsymbol{e}) \subseteq Z(\boldsymbol{s})$. Given φ, $\boldsymbol{z} \in Z^n$ is **φ-optimal for $\boldsymbol{e} \in \mathcal{E}$** if $\boldsymbol{z} \in \varphi(\boldsymbol{e})$. An SCC φ is called **efficient** if for each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, $\varphi(\boldsymbol{e}) \subseteq P(\boldsymbol{e})$. An SCC φ is called **interior** if for each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, $\varphi(\boldsymbol{e}) \subseteq \overset{\circ}{Z}(\boldsymbol{s})$.

2.2 Mechanisms

In the standard Nash implementation literature, a mechanism (game form) is defined as a pair of a set of strategy profiles and an outcome function, where the latter is a single-valued mapping from the set of strategy profiles to the set of feasible outcomes. Such a general formulation can work when implementation problems are defined in pure exchange economies as well as in production economies with known skills. However, if implementation problems are discussed in production economies with unknown skills, such a formulation is unsuitable. This is mainly due to the prominent feature of production economies with unknown skills: that the set of feasible allocations is unknown to the coordinator. This necessitates a reformulation.

Let us define the mechanisms and Nash implementation in production economies with unknown skills. First, we assume throughout the following analysis that the production function f is known and that total output after production is observable to the coordinator. This is a common feature of implementation problems in production economies for the case of not withholding,[10] since no resources can be concealed or held by agents.[11] Then, for each $\boldsymbol{s} \in \mathcal{S}^n$ and each $\boldsymbol{x} = (x_i)_{i \in N} \in [0, \bar{x}]^n$, let $w(\boldsymbol{s}, \boldsymbol{x}) \equiv f\left(\sum s_i x_i\right)$ denote total output when agents with skills $\boldsymbol{s}$ supply $\boldsymbol{x}$. Let $W(\boldsymbol{s}) \equiv \cup_{\boldsymbol{x} \in [0,\bar{x}]^n} \{w(\boldsymbol{s}, \boldsymbol{x})\}$ be the set of total outputs available under economies with $\boldsymbol{s} \in \mathcal{S}^n$ and $W \equiv \cup_{\boldsymbol{s} \in \mathcal{S}^n} W(\boldsymbol{s})$ be the universal set of total outputs with generic element w. As noted, the coordinator can observe which element of W is realized as well as the profile of supplied labor hours $\boldsymbol{x}$, but he or she cannot observe the profile of true skills $\boldsymbol{s}$.[12]

Second, as mentioned in section 1, we assume throughout the following analysis that, in the production process, the mechanism coordinator cannot affect the supply of labor hours. Therefore, the labor supplies are a part of agents' actions, rather than a part of resource allocations determined by the mechanism coordinator. Thus, the action space should be represented by the product of the space of labor supplies and the space of any other residual (non-labor) actions. That is, for each $i \in N$, let A_i, for each $i \in N$, denote the **action space** of agent i. Then, there is a space of **residual** (**non-labor**) **actions** M_i such that $A_i \equiv M_i \times [0, \bar{x}]$, and $a_i \equiv (m_i, x_i) \in A_i$ is an **action of agent** $i \in N$, where x_i represents agent i's supplied labor time and m_i her residual actions. For the sake of convenience, let us call m_i a **message of agent** i.

Given these preliminaries, let $\boldsymbol{a} \equiv (\boldsymbol{m}, \boldsymbol{x}) \in A \equiv \times_{i \in N} A_i$ denote an **action profile**, where $\boldsymbol{m} = (m_i)_{i \in N}$ is a profile of messages. For any $\boldsymbol{a} \in A$ and $i \in N$, let $\boldsymbol{a}_{-i}$ be the list $(a_j)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} A_j$ of elements of the profile $\boldsymbol{a}$ for all agents except i. Denote the set of such $\boldsymbol{a}_{-i}$ by A_{-i} for each $i \in N$. Given a list $\boldsymbol{a}_{-i} \in A_{-i}$ and an action $a_i \in A_i$ of agent i, we denote by $(a_i, \boldsymbol{a}_{-i})$ the profile consisting of these a_i and $\boldsymbol{a}_{-i}$.

A **mechanism** or **game form** γ is a pair $\gamma = (A, g)$, where $g : A \times W \rightarrow Z^n$ is the outcome function such that for each $w \in W$ and each

[10] See Hong (1995), Suh (1995), Shin and Suh (1997), Tian (1999, 2000), and Yoshihara (2000).

[11] Hong (1995) also discusses implementation problems for the case of withholding as a simple extension of the case of not withholding.

[12] Even if f, $\boldsymbol{x}$, and w are observed, the exact location of the true skill profile $\boldsymbol{s}$ cannot be known before or after the production process. For a detailed discussion on this point, see Yamada and Yoshihara (2008).

$\boldsymbol{a} = (\boldsymbol{m}, \boldsymbol{x}) \in A$, $g(\boldsymbol{a}; w) = (g_i(\boldsymbol{a}; w))_{i \in N} \in Z^n$ with $(g_{i1}(\boldsymbol{a}; w))_{i \in N} = \boldsymbol{x}$ and $\sum g_{i2}(\boldsymbol{a}; w) \leq w$. The last equation represents the property of non-authorization in allocating labor hours, in that the allocation of labor hours among agents, $(g_{i1}(\boldsymbol{a}; w))_{i \in N}$, is automatically specified by the actions of all agents' labor supplies, $\boldsymbol{x}$. The last inequality represents the feasibility of the outcome function. This property implies that given w as a total output, the aggregate of each agent's share of output, $\sum g_{i2}(\boldsymbol{a}; w)$, should not exceed the total. Denote the universal set of such game forms by Γ.

Does this definition of game forms ensure that the assigned allocation, $g(\boldsymbol{a}; w)$, is feasible for $\boldsymbol{s} \in \mathcal{S}^n$ if $\boldsymbol{s}$ is the true skill profile in the current economy? The answer is yes. Let us show this point. Given a mechanism $\gamma = (A, g) \in \Gamma$, if an action profile is specified as $\boldsymbol{a} = (\boldsymbol{m}, \boldsymbol{x}) \in A$ for each $\boldsymbol{s} \in \mathcal{S}^n$, then $w(\boldsymbol{s}, \boldsymbol{x}) \in W(\boldsymbol{s})$ is specified as the result of production, and so is $g(\boldsymbol{a}; w(\boldsymbol{s}, \boldsymbol{x}))$. As the coordinator can access the correct information about the total output $w = w(\boldsymbol{s}, \boldsymbol{x})$, he or she simply distributes such $w = w(\boldsymbol{s}, \boldsymbol{x})$ among agents according to the distribution rule described by the outcome function g. Then, the realized allocation, $g(\boldsymbol{a}; w(\boldsymbol{s}, \boldsymbol{x}))$, is always feasible for the true skill profile $\boldsymbol{s} \in \mathcal{S}^n$, even though the coordinator is ignorant of this information.

For each $\boldsymbol{s} \in \mathcal{S}^n$ and each $\boldsymbol{a}_{-i} = (\boldsymbol{m}_{-i}, \boldsymbol{x}_{-i}) \in A_{-i}$, let

$$\begin{aligned} g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}, \boldsymbol{x}_{-i})) &\equiv \{g_i(a_i, \boldsymbol{a}_{-i}; w(\boldsymbol{s}, x_i, \boldsymbol{x}_{-i})) \mid a_i = (m_i, x_i) \in A_i\}, \\ \text{where } W(\boldsymbol{s}, \boldsymbol{x}_{-i}) &\equiv \cup_{x_i' \in [0, \bar{x}]} \{w(\boldsymbol{s}, x_i', \boldsymbol{x}_{-i})\}. \end{aligned}$$

Thus, $g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}, \boldsymbol{x}_{-i}))$ is the **attainable set of agent i at $\boldsymbol{s} \in \mathcal{S}^n$**, when the profile of the other agents' actions is $\boldsymbol{a}_{-i}$. As this definition suggests, the attainable set of each agent i for a given $\boldsymbol{a}_{-i}$ may vary when the profile of skills changes.

Given $u_i \in \mathcal{U}$ and $z_i \in Z$, let $L(z_i, u_i) \equiv \{z_i' \in Z \mid u_i(z_i') \leq u_i(z_i)\}$ be **the weakly lower contour set for u_i at z_i**. Given $\gamma \in \Gamma$, for each economy $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, a (non-cooperative) game is given by $(N, \gamma, \boldsymbol{e})$. By fixing the set of players N, we simply denote a game $(N, \gamma, \boldsymbol{e})$ by $(\gamma, \boldsymbol{e})$. Given a game $(\gamma, \boldsymbol{e})$, a profile $\boldsymbol{a}^* = (\boldsymbol{m}^*, \boldsymbol{x}^*) \in A$ is a (**pure-strategy**) **Nash equilibrium of** $(\gamma, \boldsymbol{e})$ if for each $i \in N$, $g_i\left(A_i, \boldsymbol{a}^*_{-i}; W\left(\boldsymbol{s}, \boldsymbol{x}^*_{-i}\right)\right) \subseteq L\left(g_i(\boldsymbol{a}^*; w(\boldsymbol{s}, \boldsymbol{x}^*)), u_i\right)$. Let $NE(\gamma, \boldsymbol{e})$ denote the set of Nash equilibria of $(\gamma, \boldsymbol{e})$. An allocation $\boldsymbol{z} = (x_i, y_i)_{i \in N} \in Z^n$ is a **Nash equilibrium allocation of** $(\gamma, \boldsymbol{e})$ if there exists $\boldsymbol{a} = (\boldsymbol{m}, \boldsymbol{x}) \in NE(\gamma, \boldsymbol{e})$ such that $g(\boldsymbol{a}; w(\boldsymbol{s}, \boldsymbol{x})) = \boldsymbol{z}$. Let $NA(\gamma, \boldsymbol{e})$ denote the set of Nash equilibrium allocations of $(\gamma, \boldsymbol{e})$. A mechanism $\gamma \in \Gamma$

implements φ **in Nash equilibria** if for each $\boldsymbol{e} \in \mathcal{E}$, $NA(\gamma, \boldsymbol{e}) = \varphi(\boldsymbol{e})$. An $\boldsymbol{SCC}$ φ **is implementable** if there exists a mechanism $\gamma \in \Gamma$ that implements φ in Nash equilibria. Note that in this general definition of Nash implementation, the outcome function always assigns a feasible allocation. Such a feasibility condition is required in the literature on implementation theory, in the case of both abstract social choice environments such as Maskin (1999) and economic environments such as Hurwicz et al. (1995).

3 Implementation

In classical economic environments, like pure exchange economies and production economies with known skills, Nash implementation is fully characterized by Maskin monotonicity. To see this point, it is sufficient to construct a canonical-type mechanism that makes each agent announce a profile of all agents' utility functions. However, such a property no longer holds for the case of production economies with unknown skills. In that case, implementable SCCs should *not only* have a monotonicity property with respect to a specific change in the profile of utility functions, *but also* have some independence property with respect to a specific change in production skills. To examine this point precisely, we define a class of canonical mechanisms and then examine the characterization of Nash implementation by such mechanisms.[13] We also introduce a simple axiom regarding the independence of skill changes, which together with Maskin monotonicity fully characterizes Nash implementation.

A mechanism $\gamma = (A, g) \in \Gamma$ is **canonical** if for each $i \in N$, $A_i = M_i \times [0, \bar{x}] = \mathcal{U}^n \times \mathcal{S} \times Z \times [0, \bar{x}]$, with the generic element (m_i, x_i) where $m_i = (\boldsymbol{u}^i, s_i^i, z_i^i)$, and the outcome function $g : A \times W \to Z^n$ is defined as follows: for each $(\boldsymbol{m}, \boldsymbol{x}) \in A$ and each $w \in \mathbb{R}_+$, $g(\boldsymbol{m}, \boldsymbol{x}; w) = \boldsymbol{z} = (x_i, y_i)_{i \in N} \in Z^n$ satisfying $g_{i1}(\boldsymbol{m}, \boldsymbol{x}; w) = x_i$ for each $i \in N$ and $\sum g_{i2}(\boldsymbol{m}, \boldsymbol{x}; w) \leq w$. We denote by Γ_C the class of all canonical mechanisms.

Thus, in a canonical mechanism, each agent's action consists of announcing a message $m_i = (\boldsymbol{u}^i, s_i^i, z_i^i)$ and choosing his or her own supply of labor

[13] It is also possible in the same class of production economies with unknown skills to provide a full characterization of Nash implementation without any restriction on available mechanisms. In this case, the necessary and sufficient condition for Nash implementation is a variation of *Condition M* (originally introduced in Moore and Repullo (1990)), which has a highly complicated form. For this issue, see Yoshihara and Yamada (2017).

hours x_i. In each agent's message, he or she announces a profile of utility functions $\boldsymbol{u}^i$, the information about his or her own skill s_i^i, and his or her own demand for consumption z_i^i. Such a mechanism is worth calling canonical, since the message of each agent contains a profile of utility functions as in the case of the Maskin-type canonical mechanism. Such a mechanism also allows each agent to not only understate but also overstate the skill announcement.

We are interested in implementation by canonical mechanisms, which is defined as follows:

Definition 1: *An SCC φ is* canonically implementable *if there exists a canonical mechanism $\gamma = (A, g) \in \Gamma_C$ with $A_i = M_i \times [0, \bar{x}] = \mathcal{U}^n \times \mathcal{S} \times Z \times [0, \bar{x}]$ $(\forall i \in N)$ such that*:

(i) γ implements φ in Nash equilibria;
(ii) γ *is* forthright: *for each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$ and each $\boldsymbol{z} = (x_i, y_i)_{i \in N} \in \varphi(\boldsymbol{e})$, $(\boldsymbol{u}, s_i, z_i, x_i)_{i \in N} \in NE(g, \boldsymbol{e})$ and $g\left((\boldsymbol{u}, s_i, z_i, x_i)_{i \in N}; f\left(\sum s_j x_j\right)\right) = \boldsymbol{z}$.*

Denote the class of all such canonical mechanisms satisfying *forthrightness* (Definition 1-(ii))[14] by Γ_C^*.

Forthrightness is a common feature, implicitly assumed in the construction of mechanisms in most studies of Nash implementation. It requires that if a feasible allocation $\boldsymbol{z}$ is φ-optimal at the present economy $\boldsymbol{e}$, and all agents announce truthfully the information about their own skills, their own consumption bundles received in this allocation, and the true profile of utility functions, then this message profile should be a Nash equilibrium, and its equilibrium outcome should be the φ-optimal allocation, $\boldsymbol{z}$. This requirement is desirable in order to exclude the possibility of information smuggling.[15] Moreover, this requirement can eliminate an unnecessary complication in the process of computing equilibrium strategies.

We show that canonical implementation is fully characterized by two simple axioms. First, given $\boldsymbol{s} = (s_i)_{i \in N} \in \mathcal{S}^n$ and a feasible allocation $\boldsymbol{z} \in Z(\boldsymbol{s})$, let us define $Z_i(\boldsymbol{s}, \boldsymbol{x}_{-i}) \equiv \left\{z_i' \in Z \mid f\left(\sum_{j \neq i} s_j x_j + s_i x_i'\right) \geq y_i'\right\}$ as the set of feasible consumption bundles for agent i when the other agents

[14] The forthrightness condition was first introduced by Saijo, Tatamitani, and Yamato (1996) for implementation problems in pure exchange economies. Lombardi and Yoshihara (2013) then formulated this condition for implementation in abstract social choice problems.

[15] For a detailed explanation about information smuggling and about how forthrightness can exclude this problem, see Lombardi and Yoshihara (2013).

supply $\boldsymbol{x}_{-i}$ at $\boldsymbol{s} \in \mathcal{S}^n$. Moreover, given an economy $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$ and a feasible allocation $\boldsymbol{z} \in Z(\boldsymbol{s})$, let $L(\boldsymbol{z}, u_i; \boldsymbol{s}) \equiv L(z_i, u_i) \cap Z_i(\boldsymbol{s}, \boldsymbol{x}_{-i})$. Then, the standard axiom of Maskin monotonicity can be introduced in production economies with unknown skills as follows:

Monotonicity (M): *For each* $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$, $\boldsymbol{e}' = (\boldsymbol{u}', \boldsymbol{s}) \in \mathcal{E}$ *and each* $\boldsymbol{z} \in \varphi(\boldsymbol{e})$, *if* $L(\boldsymbol{z}, u_i; \boldsymbol{s}) \subseteq L(\boldsymbol{z}, u_i'; \boldsymbol{s})$ *for each* $i \in N$, *then* $\boldsymbol{z} \in \varphi(\boldsymbol{u}', \boldsymbol{s})$.

This condition is a natural reformulation of *Maskin monotonicity* [Maskin (1999)] suited to production economies considered here. Remember that in pure exchange economies, the *monotonic transformation of agent i's utility function* presumed by Maskin monotonicity takes place within a restricted consumption space, where the restriction is given by the aggregate endowments of commodities as the upper bound of feasible consumption bundles.[16] A corresponding constraint on the consumption space in the case of production economies is given by the set of feasible consumption bundles, $Z_i(\boldsymbol{s}, \boldsymbol{x}_{-i})$. Thus, in production economies, the monotonic transformation of agent i's utility function should be represented by $L(\boldsymbol{z}, u_i; \boldsymbol{s}) \subseteq L(\boldsymbol{z}, u_i'; \boldsymbol{s})$ for the definition of Maskin monotonicity.

The second axiom is relevant to the change in individual skills.

Non-manipulability of Unused Skills (NUS): *For each* $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$ *and each* $\boldsymbol{z} \in \varphi(\boldsymbol{e})$, *for each* $\boldsymbol{e}' = (\boldsymbol{u}, \boldsymbol{s}') \in \mathcal{E}$ *where* $s_j' = s_j$ *for each* $j \in N$ *with* $x_j > 0$, *if* $\boldsymbol{z} \in P(\boldsymbol{e}') \setminus \varphi(\boldsymbol{e}')$, *then for each* $i \in N$ *with* $s_i' \neq s_i$, *there is no* $z_i' \in Z$ *such that* $(z_i', \boldsymbol{z}_{-i}) \in \varphi(\boldsymbol{e}')$.

That is, suppose that $\boldsymbol{z}$ is a φ-optimal allocation at the present economy $\boldsymbol{e}$, where an agent $i \in N$ supplies no labor, $x_i = 0$. Consider that the economy changes from the present $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$ to $\boldsymbol{e}' = (\boldsymbol{u}, \boldsymbol{s}')$, in that only agent i's skill is changed from s_i to $s_i' (\neq s_i)$. Moreover, suppose that $\boldsymbol{z}$ is still efficient, but no longer φ-optimal at the new economy $\boldsymbol{e}'$. Then, **NUS** requires that no other φ-optimal allocation can be found just by replacing agent i's consumption bundle from z_i.

Insert Figure 1.

To see this axiom intuitively, let us consider an example of production economies with two agents. Suppose that $N = \{1, 2\}$ with $s_1 = 1 = s_2 < s_1'$,

[16] Moreover, in the abstract social choice environments, the monotone transformation takes place within the set of feasible alternatives.

and the production function is given by $f(x) = x$ for any $x \geqq 0$. Moreover, consider a feasible allocation (z_1, z_2) with $x_1 = 0 < x_2$, $0 < y_1 < y_2$, and $f(s_2 x_2) = x_2 = y_1 + y_2$. Finally, let (u_1, u_2) be represented by the indifference curves given in *Figure* 1*a*. Then, the allocation (z_1, z_2) is Pareto efficient for an economy $\boldsymbol{e} = ((u_1, u_2), (s_1, s_2))$. Suppose that (z_1, z_2) is φ-optimal at the economy $\boldsymbol{e} = ((u_1, u_2), (s_1, s_2))$. Moreover, suppose that the economy changes from $\boldsymbol{e} = ((u_1, u_2), (s_1, s_2))$ to $\boldsymbol{e}' = ((u_1, u_2), (s'_1, s_2))$. Then, though the allocation (z_1, z_2) is still Pareto efficient for the new economy $\boldsymbol{e}'$, in *Figure* 1*a*, let us assume that (z_1, z_2) is no longer φ-optimal. In this case, given that u_1 is strictly quasi-concave in *Figure* 1*a*, there cannot exist any other Pareto efficient allocation with z_2 as agent 2's consumption bundle for $\boldsymbol{e}'$. Therefore, no φ-optimal allocation can be found at $\boldsymbol{e}'$ just by replacing agent 1's consumption bundle from z_1. Thus, the case of *Figure* 1*a* describes how the *SCC* φ satisfies **NUS**.

In contrast, consider another type of preference profile (u_1, u_2) given the same feasible allocation (z_1, z_2) for the same skill profile (s_1, s_2) and the same production function $f(x) = x$. This preference profile is represented by the indifference curves given in *Figure* 1*b*. As seen in *Figure* 1*b*, (z_1, z_2) is Pareto efficient for an economy $\boldsymbol{e} = ((u_1, u_2), (s_1, s_2))$, and so let us assume that (z_1, z_2) is φ-optimal at $\boldsymbol{e}$. Then, as in the case of *Figure* 1*a*, let us assume that the economy changes from $\boldsymbol{e} = ((u_1, u_2), (s_1, s_2))$ to $\boldsymbol{e}' = ((u_1, u_2), (s'_1, s_2))$, where $s_1 = 1 = s_2 < s'_1$, in *Figure* 1*b*. As before, while the allocation (z_1, z_2) is still Pareto efficient for the new economy $\boldsymbol{e}'$, in *Figure* 1*b*, let us assume that (z_1, z_2) is no longer φ-optimal at this new economy. However, in this case, as u_1 is not strictly quasi-concave in *Figure* 1*b*, there is another Pareto efficient allocation (z'_1, z_2) for $\boldsymbol{e}'$. Thus, if this allocation is also φ-optimal at $\boldsymbol{e}'$, then the *SCC* φ does not satisfy **NUS** in *Figure* 1*b*.

Note that **NUS** has an independence property in terms of particular types of skill changes in a quite weak sense. Indeed, if the allocation (z_1, z_2) in the above *Figure* 1 is φ-optimal at $\boldsymbol{e}'$, then the requirement of **NUS** is trivially satisfied by this allocation, which represents the independence feature of this *SCC* φ with respect to the change of skills specified above.[17]

We show that an *SCC* is canonically implementable only if it satisfies **M**

[17] By presuming exactly the same type of skill changes as that of **NUS**, the *independence of unused skills* (**IUS**) axiom introduced by Yamada and Yoshihara (2007) straighthorwardly requires that the φ-optimal allocation at the economy $\boldsymbol{e}$ should also be φ-optimal at the new economy $\boldsymbol{e}'$. **NUS** is much weaker than **IUS**.

and **NUS**.

Theorem 1: *If an efficient SCC* φ *is* canonically implementable*, then* φ *satisfies* **M** *and* **NUS**.

Proof. Let φ be a canonically implementable *SCC*. Then, there exists a canonical mechanism $\gamma = (A, g)$ such that for any $\boldsymbol{e} = (u_i, s_i)_{i \in N} \in \mathcal{E}$, $NA(\gamma, \boldsymbol{e}) = \varphi(\boldsymbol{e})$. If $\boldsymbol{z} \in \varphi(\boldsymbol{e})$ and $\boldsymbol{a} = (\boldsymbol{u}, s_j, z_j, x_j)_{j \in N} \in NE(\gamma, \boldsymbol{e})$ with $g(\boldsymbol{a}; f(\sum s_j x_j)) = \boldsymbol{z}$, then $g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}, \boldsymbol{x}_{-i})) \subseteq L(\boldsymbol{z}, u_i; \boldsymbol{s})$ for each $i \in N$, from the definition of the mechanism. Let $\boldsymbol{e}' = (u_i', s_i)_{i \in N} \in \mathcal{E}$ such that $L(\boldsymbol{z}, u_i; \boldsymbol{s}) \subseteq L(\boldsymbol{z}, u_i'; \boldsymbol{s})$ for each $i \in N$. Then, $g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}, \boldsymbol{x}_{-i})) \subseteq L(\boldsymbol{z}, u_i'; \boldsymbol{s})$ for each $i \in N$. Thus, $\boldsymbol{a} \in NE(\gamma, \boldsymbol{e}')$ holds, which implies $\boldsymbol{z} \in \varphi(\boldsymbol{e}')$. Thus, the necessity of **M** is verified.

Let us show the necessity of **NUS**. For each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$, $\boldsymbol{e}' = (u_i, s_i')_{i \in N} \in \mathcal{E}$, and each $\boldsymbol{z} \in \varphi(\boldsymbol{e})$, let $s_j = s_j'$ for each $j \in N$ with $x_j > 0$. Moreover, let $\boldsymbol{z} \in P(\boldsymbol{e}') \backslash \varphi(\boldsymbol{e}')$. Suppose for some $i \in N$ with $s_i' \neq s_i$ and $x_i = 0$, there exists $z_i' \in Z$ with $x_i' > 0$ such that $\boldsymbol{z}' \equiv (z_i', \boldsymbol{z}_{-i}) \in \varphi(\boldsymbol{e}')$. Since $\boldsymbol{z} \in P(\boldsymbol{e}')$ and $(z_i', \boldsymbol{z}_{-i}) \in \varphi(\boldsymbol{e}') \subseteq P(\boldsymbol{e}')$, $u_i(z_i) = u_i(z_i')$ holds. According to the canonical implementability of φ, there exist $\boldsymbol{a} = (\boldsymbol{u}, s_j, z_j, x_j)_{j \in N} \in NE(\gamma, \boldsymbol{e})$ and $\boldsymbol{a}' = \left(\boldsymbol{u}, s_j', z_j', x_j'\right)_{j \in N} \in NE(\gamma, \boldsymbol{e}')$ such that $g(\boldsymbol{a}; w(\boldsymbol{s}, \boldsymbol{x})) = \boldsymbol{z}$ and $g(\boldsymbol{a}'; w(\boldsymbol{s}', \boldsymbol{x}')) = \boldsymbol{z}'$. From the definition of Nash equilibrium, it follows that for any $j \neq i$, $g_j(A_j, \boldsymbol{a}_{-j}; W(\boldsymbol{s}, \boldsymbol{x}_{-j})) \subseteq L(z_j, u_j)$ and $g_j\left(A_j, \boldsymbol{a}'_{-j}; W\left(\boldsymbol{s}', \boldsymbol{x}'_{-j}\right)\right) \subseteq L(z_j, u_j)$; and for i, $g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}, \boldsymbol{x}_{-i})) \subseteq L(z_i, u_i)$ and $g_i\left(A_i, \boldsymbol{a}'_{-i}; W\left(\boldsymbol{s}', \boldsymbol{x}'_{-i}\right)\right) \subseteq L(z_i', u_i)$. Note that for any $j \neq i$, $a_j = (\boldsymbol{u}, s_j, z_j, x_j) = \left(\boldsymbol{u}, s_j', z_j', x_j'\right) = a_j'$ holds. Hence, $g_i\left(A_i, \boldsymbol{a}'_{-i}; W\left(\boldsymbol{s}', \boldsymbol{x}'_{-i}\right)\right) = g_i(A_i, \boldsymbol{a}_{-i}; W(\boldsymbol{s}', \boldsymbol{x}_{-i})) \subseteq L(z_i', u_i) = L(z_i, u_i)$ holds. Moreover, since $W(\boldsymbol{s}, \boldsymbol{x}_{-j}) = \left\{f\left(\sum_{k \neq i,j} s_k x_k + s_j x_j^*\right) \mid x_j^* \in [0, \overline{x}]\right\} = W\left(\boldsymbol{s}', x_i, \boldsymbol{x}'_{-i,j}\right)$, $g_j\left(A_j, a_i, \boldsymbol{a}'_{-i,j}; W\left(\boldsymbol{s}', x_i, \boldsymbol{x}'_{-i,j}\right)\right) = g_j(A_j, \boldsymbol{a}_{-j}; W(\boldsymbol{s}, \boldsymbol{x}_{-j})) \subseteq L(z_j, u_j)$ holds for any $j \neq i$. Therefore, $\left(a_i, \boldsymbol{a}'_{-i}\right) = \boldsymbol{a} \in NE(\gamma, \boldsymbol{e}')$, and $g\left(\boldsymbol{a}; f\left(\sum_{j \neq i} s_j' x_j + s_i' x_i\right)\right) = g\left(\boldsymbol{a}; f\left(\sum_{j \neq i} s_j x_j\right)\right) = g\left(\boldsymbol{a}; f\left(\sum_{j \neq i} s_j x_j + s_i x_i\right)\right) = \boldsymbol{z} \in NA(\gamma, \boldsymbol{e}')$, which is a contradiction from the implementability of φ, since $\boldsymbol{z} \notin \varphi(\boldsymbol{e}')$. Thus, φ satisfies **NUS**. ■

Next, we show that under **Assumption 1**, an interior and efficient *SCC* is canonically implementable if it satisfies **M** and **NUS**.

Theorem 2: *Let Assumption 1 hold and* $n \geq 3$*. Then, if an interior and efficient SCC* φ *satisfies* **M** *and* **NUS**, φ *is* canonically implementable.

Although the proof of Theorem 2 is provided in the Appendix, here we briefly explain the canonical mechanism constructed in this proof. According to the definition of canonical mechanisms, the constructed mechanism in the proof of Theorem 2, say $\gamma^* = (A^*, g^*) \in \Gamma_C^*$, also has the action set $A_i^* \equiv M_i \times [0, \bar{x}] = \mathcal{U}^n \times \mathcal{S} \times Z \times [0, \bar{x}]$ for each $i \in N$, and let $\boldsymbol{a} = \left((\boldsymbol{u}^i)_{i \in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \boldsymbol{x} \right) \equiv (\boldsymbol{u}^i, \sigma_i, z_i, x_i)_{i \in N}$ denote the generic action profile in this mechanism. Let an action profile $\boldsymbol{a} = \left((\boldsymbol{u}^i)_{i \in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \boldsymbol{x} \right) \in A^*$ with $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$ be called **φ-consistent** (*resp.* **P-consistent**) if for some $\boldsymbol{u} \in \mathcal{U}^n$, $\boldsymbol{u}^i = \boldsymbol{u}$ for each $i \in N$ and $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$ (*resp.* $(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{u}, \boldsymbol{\sigma})$).

For each $\boldsymbol{a} = \left((\boldsymbol{u}^i)_{i \in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \boldsymbol{x} \right) \in A^*$ and each produced output $w = w(\boldsymbol{s}, \boldsymbol{x}) \in W$, γ^* generally works as follows: First, g^* computes the amount $f\left(\sum \sigma_k x_k\right)$ and compares this with w. Let $f\left(\sum \sigma_k x_k\right) = w$. Then, if $\boldsymbol{a}$ is φ-consistent, g^* distributes w in accordance with $\boldsymbol{y}$, so that $g^*(\boldsymbol{a}; w) = (\boldsymbol{x}, \boldsymbol{y})$ (**Rule 1-1-a**). If $\boldsymbol{a}$ is not φ-consistent but P-consistent, then g^* punishes everyone, meaning that $g^*(\boldsymbol{a}; w) = (\boldsymbol{x}, \mathbf{0})$ (**Rule 1-1-b**).

Let there be a unique potential deviator, say j, under $f\left(\sum \sigma_k x_k\right) = w$, in that $\boldsymbol{u}^i = \boldsymbol{u} \neq \boldsymbol{u}^j$ for each $i \neq j$, and $(\boldsymbol{x}, \boldsymbol{y}) \notin \varphi(\boldsymbol{u}^j, \boldsymbol{\sigma})$. In this situation, there may be two cases. First, let j have the property that there exists $z_j^\circ = \left(x_j^\circ, y_j^\circ\right) \in Z$ such that $\left(\left(x_j^\circ, \boldsymbol{x}_{-i}\right), \left(y_j^\circ, \boldsymbol{y}_{-i}\right) \right) \in \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$. Let us call such j a *potential deviator*. In this case, the mechanism would conclude that j might deviate from $\left(\boldsymbol{u}, \sigma_j, z_j^\circ, x_j^\circ\right)$ to the current action $(\boldsymbol{u}^j, \sigma_j, z_j, x_j)$. Thus, j would be punished in such a way that $g_j^*(\boldsymbol{a}; w) = \left(x_j, y_j''\right)$ is placed within the lower contour set of $\left(x_j^\circ, y_j^\circ\right)$ at $(\boldsymbol{u}, \boldsymbol{\sigma})$ or the budget sets defined by the efficiency prices for $\left(\boldsymbol{z}_{-j}, \left(x_j^\circ, y_j^\circ\right)\right)$ at $(\boldsymbol{u}, \boldsymbol{\sigma})$ (**Rule 1-2-a**). Second, j may not be identified as a potential deviator in the sense specified in **Rule 1-2-a** above, and there may exist $\left(x_j', y_j'\right) \in Z$ such that $\left(\left(x_j', \boldsymbol{x}_{-i}\right), \left(y_j', \boldsymbol{y}_{-i}\right) \right) \in P(\boldsymbol{u}^j, \boldsymbol{\sigma})$. In this case, $g_j^*(\boldsymbol{a}; w) = (x_j, w)$ whenever $x_j > 0$ and $y_j > w$; and otherwise, $g_j^*(\boldsymbol{a}; w) = (x_j, 0)$ (**Rule 1-2-b**).

For any other case under $f\left(\sum \sigma_k x_k\right) = w$, g^* would pick up the agent, say j, who supplies the maximal labor hours among all agents. Then, it would assign the total output produced to him or her: $g_j^*(\boldsymbol{a}; w) = (x_j, w)$ (**Rule 1-3**).

Second, let $f\left(\sum \sigma_k x_k\right) \neq w$. In this case, if there exists an agent, say j, who provides zero labor hours and announces the highest skill level among all agents, then g^* would assign the total output produced to him or her: $g_j^*(\boldsymbol{a}; w) = (0, w)$. If there is no such agent, then $g^*(\boldsymbol{a}; w) = (\boldsymbol{x}, \mathbf{0})$ (**Rule 2**).

Now, let us briefly explain how γ^* induces the true revelation of skills, at least for working agents (A), and how it attains desirable allocations (B):

(A) Let $\boldsymbol{s}$ be the true profile of skills and $f\left(\sum s_k x_k\right)$ be total produced output. First, if $f\left(\sum \sigma_k x_k\right) \neq f\left(\sum s_k x_k\right)$, then clearly $\boldsymbol{\sigma} \neq \boldsymbol{s}$, and at least one agent, say $j \in N$ whose labor supply is positive, has misrepresented his or her skill, $\sigma_j \neq s_j$. According to the above explanation about g^*, such an agent is punished with $g_j^*\left(\boldsymbol{a}; f\left(\sum s_k x_k\right)\right) = (x_j, 0)$ in **Rule 2**. Then, one such agent may be better off by shifting to an alternative action in which either (i) he or she supplies zero labor hours and falsely announces having the highest skill level among all agents (to induce **Rule 2**) or (ii) he or she supplies the maximal labor hours among all agents and truthfully announces his or her own skill (to induce **Rule 1-3**). In either case, the agent in question receives the total output produced. Thus, the action profile with $f\left(\sum \sigma_k x_k\right) \neq f\left(\sum s_k x_k\right)$ cannot correspond to an equilibrium.

Second, suppose $f\left(\sum \sigma_k x_k\right) = f\left(\sum s_k x_k\right)$ but $\boldsymbol{\sigma} \neq \boldsymbol{s}$. Then, *either* at least two agents have misrepresented their skills while supplying positive amounts of labor *or* someone, say j, supplies no labor hours while misrepresenting his or her skill. In the former case, among such misrepresenting agents, there exists an agent, say j, who can switch from $x_j > 0$ to $x_j' = 0$, while announcing a higher number σ_j' than any other in $\boldsymbol{\sigma}_{-j}$ to induce $f\left(\sigma_j' x_j' + \sum_{i \neq j} \sigma_i x_i\right) \neq f\left(\sum_{i \neq j} s_i x_i\right)$ (**Rule 2**). Then, j may be better off by receiving $\left(0, f\left(\sum_{i \neq j} s_i x_i\right)\right)$. Thus, this case cannot correspond to an equilibrium. In other words, $f\left(\sum \sigma_k x_k\right) = f\left(\sum s_k x_k\right)$ and $\sigma_i = s_i$ for any $i \in N$ with $x_i > 0$ must hold in any equilibrium, as confirmed by Lemma A1 in the Appendix.

(B) Let $f\left(\sum \sigma_k x_k\right) = f\left(\sum s_k x_k\right)$ and $\sigma_i = s_i$ for any $i \in N$ with $x_i > 0$ for the action profile $\boldsymbol{a}$. It can be shown that unless $\boldsymbol{a}$ is φ-consistent, it cannot be an equilibrium. Therefore, let $\boldsymbol{a}$ be φ-consistent in **Rule 1-1-a** and the corresponding outcome $(\boldsymbol{x}, \boldsymbol{y})$ is φ-optimal for some economy. However, $(\boldsymbol{x}, \boldsymbol{y})$ may not be φ-optimal for the actual economy, because *either* $\left(\boldsymbol{u}^i\right)_{i \in N}$ is not equal to the true profile of utility functions *or* there exists a "non-working" agent j who misrepresents his or her true skill. In the former case, since φ satisfies **M**, there exists an agent whose lower contour set of his or her true utility function at this allocation does not contain the lower contour set of his or her announced utility function at this allocation. Then, this agent can change his or her strategy to induce **Rule 1-2-a** and be better off by this deviation. In the latter case, since φ satisfies **NUS**, the misrepresenting "non-working" agent should not be a potential deviator in the sense specified in **Rule 1-2-a** above. Therefore, he or she can change his or her action to induce **Rule 1-2-b**, where he or she announces the true skill level and a suitably changed profile of utility functions $\boldsymbol{u}'^j$ in order to find some $\left(x'_j, y'_j\right) \in Z$ such that $\left(\left(x'_j, \boldsymbol{x}_{-i}\right), \left(y'_j, \boldsymbol{y}_{-i}\right)\right) \in P\left(\boldsymbol{u}'^j, \boldsymbol{s}\right)$, and supplies a small positive amount of labor to make him- or herself better off. In summary, if $(\boldsymbol{x},\boldsymbol{y})$ is an equilibrium allocation, it should be φ-optimal for the actual economy.

We are now ready to discuss the full characterization of canonical implementation.

Corollary 1: *Let Assumption 1 hold and $n \geq 3$. Then, an interior and efficient SCC φ is* canonically implementable *if and only if φ satisfies* **M** *and* **NUS**.

Although canonical implementation is not equivalent to Nash implementation, we show in the next section that these are equivalent when the production function f is strictly concave, since in such a case **NUS** is vacuously satisfied.

4 Applications

It is well known that in production economies with unequal skills, the *no-envy and efficient solution* [Foley (1967)] is not well defined. Thus, we consider a weaker version of the no-envy principle to define equitable *SCC*s. One such example is the *equal-opportunity-for-budget-set* (**EOB**) principle that can be formulated as follows. Given $\boldsymbol{s} \in \mathcal{S}^n$, $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in Z^n$, and $p \in \Delta$, let $B\left(p, s_i, z_i\right) \equiv \{z'_i \in Z \mid p_y y'_i - p_x s_i x'_i \leq p_y y_i - p_x s_i x_i\}$. Then:

Set-inclusion Undomination (SIU):[18] *For each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$ and each $\boldsymbol{z} \in \varphi(\boldsymbol{e})$, there exists $p \in \Delta^P(\boldsymbol{e}, \boldsymbol{z})$ such that for each $i, j \in N$, neither $B\left(p, s_i, z_i\right) \subsetneq B\left(p, s_j, z_j\right)$ nor $B\left(p, s_i, z_i\right) \supsetneq B\left(p, s_j, z_j\right)$.*

Any *SCC* satisfying the no-envy principle also satisfies **SIU**. In addition, a weaker version of the no-envy and efficient solution, the *$\widetilde{u}$-reference welfare equivalent budget solution* [Fleurbaey and Maniquet (1996)], satisfies **SIU**.

[18] Van Parijs (1995) formulated the EOB principle as **undominated diversity** [Parijs (1995)], which is stronger than SIU.

Definition 2: *An SCC is the* $\widetilde{u}$-reference welfare equivalent budget solution $\varphi^{\widetilde{u}\text{-}RWEB}$ *if for each* $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, $\boldsymbol{z} \in \varphi^{\widetilde{u}\text{-}RWEB}(\boldsymbol{e})$ *implies that* $\boldsymbol{z} \in P(\boldsymbol{e})$; *and there exists* $p = (p_x, p_y) \in \Delta^P(\boldsymbol{e}, \boldsymbol{z})$ *for* $\boldsymbol{z}$ *at* $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$ *such that for any* $i, j \in N$, $\max_{z' \in B(p, s_i, z_i)} \widetilde{u}(z') = \max_{z' \in B(p, s_j, z_j)} \widetilde{u}(z')$.

Interestingly, most equitable SCCs including $\varphi^{\widetilde{u}\text{-}RWEB}$ are implementable. To see this, let φ be an SCC. This φ meets *non-discrimination* if, for any $(\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, any $\boldsymbol{z} \in \varphi(\boldsymbol{u}, \boldsymbol{s})$, and any $\boldsymbol{z}' \in P(\boldsymbol{u}, \boldsymbol{s})$ such that $u_i(z_i) = u_i(z_i')$ for each $i \in N$, $\boldsymbol{z}' \in \varphi(\boldsymbol{u}, \boldsymbol{s})$ holds. We have:

Lemma 1: *If an efficient SCC satisfies* non-discrimination, *then it satisfies* **NUS**.

Proof. Let φ be an efficient SCC that does not satisfy **NUS**. For each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$, $\boldsymbol{e}' = (u_i, s_i')_{i \in N} \in \mathcal{E}$, and each $\boldsymbol{z} \in \varphi(\boldsymbol{e})$, let $s_j = s_j'$ for each $j \in N$ with $x_j > 0$. Moreover, let $\boldsymbol{z} \in P(\boldsymbol{e}') \backslash \varphi(\boldsymbol{e}')$. Suppose for some $i \in N$ with $s_i' \neq s_i$, there exists $z_i' \in Z$ such that $(z_i', \boldsymbol{z}_{-i}) \in \varphi(\boldsymbol{e}')$. Since $\boldsymbol{z} \in P(\boldsymbol{e}')$ and $(z_i', \boldsymbol{z}_{-i}) \in \varphi(\boldsymbol{e}') \subseteq P(\boldsymbol{e}')$, $u_i(z_i) = u_i(z_i')$ holds. Thus, since φ satisfies non-discrimination, $\boldsymbol{z} \in \varphi(\boldsymbol{e}')$, which is a contradiction. ■

Note that an efficient SCC may not satisfy non-discrimination, but could satisfy **NUS**. For instance, the *proportional solution* [Roemer and Silvestre (1993)] is such an SCC.

Corollary 2: *Any interior and efficient SCC satisfying* non-discrimination *is* canonically implementable *if and only if it satisfies* **M**.

Thus, an equitable solution satisfying non-discrimination is canonically implementable if it satisfies **M**. There are many such SCCs, including $\varphi^{\widetilde{u}\text{-}RWEB}$.

Is there an SCC that satisfies **M** but not **NUS**? First, let us consider the case that the production function f is strictly concave. Then:

Lemma 2: *Let* f *be strictly concave. Then, any efficient SCC satisfies* **NUS**.

Proof. Take any $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$, $\boldsymbol{e}' = (\boldsymbol{u}, \boldsymbol{s}') \in \mathcal{E}$ such that (i) $s_j = s_j'$ for each $j \in N \backslash \{1\}$, and $s_1 \neq s_1'$. Let $\boldsymbol{z} \in \varphi(\boldsymbol{e})$ with $z_1 = (0, y_1)$. Suppose $\boldsymbol{z} \in P(\boldsymbol{e}') \backslash \varphi(\boldsymbol{e}')$. Then, given $\boldsymbol{z}_{-1}$, there is no other consumption bundle z_1' such that $(z_1', \boldsymbol{z}_{-1}) \in P(\boldsymbol{e}')$ holds. This fact is because f is strictly concave, meaning that for any efficiency price $p \in \Delta^P(\boldsymbol{e}', \boldsymbol{z})$, z_1 is the unique intersection point of $\partial B(p, s_1', z_1)$ and the set $\left\{(x,y) \in Z \mid y = f\left(\sum_{j \neq 1} s_j x_j + s_1' x\right) - \sum_{j \neq 1} y_j\right\}$. Thus, φ satisfies **NUS**. ■

Corollary 3: *Let f be strictly concave. Then, an interior and efficient SCC is canonically implementable if and only if it satisfies* **M**.

Second, consider the case that the production function f is not strictly concave. In this case, we can find an interior and efficient SCC that satisfies **M** but not **NUS**. Given $\boldsymbol{s} \in \mathcal{S}^n$, there exists an agent whose skill level is the lowest within the population at $\boldsymbol{s}$. Denote such an agent at $\boldsymbol{s}$ by $i(\boldsymbol{s})$. Then:

Definition 3: *An SCC is the* maximal workfare solution φ^{WF} *if for each $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$, $\boldsymbol{z} \in \varphi^{WF}(\boldsymbol{e})$ implies that there exists an efficiency price $p \in \Delta^P(\boldsymbol{e}, \boldsymbol{z})$ such that $\boldsymbol{z} \in \arg\max_{z' \in P(e)} p_y y'_{i(s)} - p_x s_{i(s)} x'_{i(s)}$ and there is no $z''_{i(s)} \in Z$ with $x''_{i(s)} > x_{i(s)}$ and $\left(z''_{i(s)}, \boldsymbol{z}_{-i(s)}\right) \in P(\boldsymbol{e})$.*

To see an implication of this solution, let us assume that f is linear. Then, let p^* be the efficiency price of any Pareto-efficient allocation, which has the property that $\frac{p_x^*}{p_y^*} = \frac{f(x)}{x}$ holds for any $x > 0$. Given this p^*, if $\left(\left(x_{i(s)}, y_{i(s)}\right), \boldsymbol{z}_{-i(s)}\right), \left(\left(0, y^*_{i(s)}(\boldsymbol{e})\right), \boldsymbol{z}_{-i(s)}\right) \in P(\boldsymbol{e})$ with $y_{i(s)} = y^*_{i(s)}(\boldsymbol{e}) + \frac{p_x^*}{p_y^*} s_{i(s)} x_{i(s)}$ are such that $u_{i(s)}\left(x_{i(s)}, y_{i(s)}\right) = u_{i(s)}\left(0, y^*_{i(s)}(\boldsymbol{e})\right)$, then φ^{WF} never selects $\left(\left(0, y^*_{i(s)}(\boldsymbol{e})\right), \boldsymbol{z}_{-i(s)}\right)$, since the welfare payment $y^*_{i(s)}(\boldsymbol{e})$ via φ^{WF} is to encourage the lowest skill agent to work. In other words, φ^{WF} provides the lowest skilled agents with the maximal welfare payment if and only if they work as much as possible, which is the reason why we call φ^{WF} the *workfare* solution.

It is easy to see that φ^{WF} does not satisfy non-discrimination. Moreover:

Lemma 3: *Let f be linear. Then, φ^{WF} is an interior and efficient SCC that satisfies* **M***, but does not satisfy* **NUS**.

Proof. It is easy to check that φ^{WF} satisfies **M**. Let us check that φ^{WF} does not satisfy **NUS**. Take $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s})$, $\boldsymbol{e}' = (\boldsymbol{u}, \boldsymbol{s}') \in \mathcal{E}$ such that (i) $s_j = s_j'$ and $s_1 < s_1' < s_j$ for each $j \in N \setminus \{1\}$; (ii) there is $\boldsymbol{z} \in P(\boldsymbol{e})$ such that $\boldsymbol{z} = \arg\max_{z' \in P(e)} p_y y_1' - p_x s_1 x_1'$ and $z_1 = (0, y_1)$, where $\frac{p_y}{p_x} = \frac{f(x)}{x}$ holds for any $x > 0$, by linearity of f. Without loss of generality, let this

z_1 be the unique solution to maximize $u_1(z)$ subject to $z \in B(p, s_1, z_1)$. Such uniqueness is ensured if s_1 is sufficiently small. Furthermore, (iii) let us assume under $\boldsymbol{e}'$ that u_1 has a quasi-linear form $u_1(x, y) \equiv v_1(x) + y$ with a property that there is an interval $[0, x_1']$ such that for any $x \in [0, x_1']$, $\left(x, y_1 + \frac{p_x}{p_y} s_1' x\right) \in \arg\max_{z \in B\left(p, s_1', z_1\right)} u_1(z)$.

By definition, $\boldsymbol{z} \in \varphi^{WF}(\boldsymbol{e})$. Note $\boldsymbol{z} \in P(\boldsymbol{e}')$. Then, $z_1 \in \arg\max_{z'' \in P(e')} p_y y_1'' - p_x s_1' x_1''$. To see this, let us take any $z_1^* \equiv \left(x_1^*, y_1^* + \frac{p_x}{p_y} s_1' x_1^*\right) \in \arg\max_{z'' \in P(e')} p_y y_1'' - p_x s_1' x_1''$, and suppose $y_1^* > y_1$. Let $\boldsymbol{z}^* \in P(\boldsymbol{e}')$, whose first component is z_1^*. Then, $\left((0, y_1^*), \boldsymbol{z}_{-1}^*\right) \in P(\boldsymbol{e}')$ from the quasi-linearity of u_1. Moreover, $\left((0, y_1^*), \boldsymbol{z}_{-1}^*\right)$ is also Pareto-efficient for $\boldsymbol{e}$, which implies that $\boldsymbol{z} \notin \varphi^{WF}(\boldsymbol{e})$, and thus is a contradiction. Therefore, $y_1^* = y_1$ holds, which implies the desired result.

From (iii), $z_1' \equiv \left(x_1', y_1 + \frac{p_x}{p_y} s_1' x_1'\right) \in \arg\max_{z'' \in P(e')} p_y y_1'' - p_x s_1' x_1''$, and $x_1' > 0$, which implies that $\boldsymbol{z} \notin \varphi^{WF}(\boldsymbol{e}')$, whereas $(z_1', \boldsymbol{z}_{-1}) \in \varphi^{WF}(\boldsymbol{e}')$. Thus, φ^{WF} does not satisfy **NUS**. ■

Corollary 4: *Let f be linear. Then, φ^{WF} is not canonically implementable.*

Note that φ^{WF} is Nash-implementable if skills are not private information, since φ^{WF} satisfies Maskin monotonicity.

Thus, Corollaries 1, 3, and 4 suggest that within the class of interior and efficient SCCs, canonical implementation is equivalent to Nash implementation whenever the production function is strictly concave, but they are not equivalent when the production function is linear.

5 Concluding Remarks

We characterized Nash implementation in production economies with unequal labor skills. In particular, we characterized the class of interior and efficient SCCs that are Nash-implementable by canonical mechanisms. The axioms for this full characterization, **M** and **NUS**, are simple and easy to test. The restriction on the available mechanisms is merely subtle, in that the Nash implementability of interior and efficient SCCs with and without this restriction are equivalent whenever the production function is strictly concave.

Note that the results would change if a further restriction were introduced to the available class of mechanisms. If the only available class were simple mechanisms, where each agent is required to announce a price vector rather than a profile of utility functions, then the second main result presented in this paper would no longer hold. That is, when utility functions change Maskin monotonicity must be replaced by a much stronger monotonicity condition and **NUS** should also be replaced by a stronger variant. Yamada and Yoshihara (2007) and Yoshihara and Yamada (2010) developed this line of research.

6 Appendix

As a preliminary step, we construct two auxiliary functions, which are used in the proofs of Theorem 2. Given $\boldsymbol{x} \in [0, \bar{x}]^n$ and $i \in N$, let

$$\pi(\boldsymbol{x}_{-i}) \equiv \begin{cases} \max_{j \neq i \text{ s.t. } x_j < \bar{x}} \frac{x_j + \bar{x}}{2} & \text{if there exists } j \in N \backslash \{i\} \text{ such that } x_j < \bar{x}, \\ \bar{x} & \text{otherwise.} \end{cases}$$

Let the notation $\sigma_i \in \mathcal{S}$ represent the announced skill of agent i, which is not necessarily identical to a truthful skill $s_i \in \mathcal{S}$ of agent i, and $\boldsymbol{\sigma} = (\sigma_i)_{i \in N} \in \mathcal{S}^n$ represent a profile of the announced skills of all agents. Then,

- Let $d^{\boldsymbol{y}}$ be such that for each $w \in \mathbb{R}_+$, each $(\boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}^n \times [0, \bar{x}]^n \times \mathbb{R}^n_+$, and each $i \in N$,
$$d_i^{\boldsymbol{y}}(\boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y}, w) = \begin{cases} w & \text{if } x_i = \pi(\boldsymbol{x}_{-i}) < \bar{x} \text{ or } x_i < \pi(\boldsymbol{x}_{-i}) = \bar{x}, \text{ and } y_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$
- Let $d^{\boldsymbol{\sigma}}$ be such that for each $w \in \mathbb{R}_+$, each $(\boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}^n \times [0, \bar{x}]^n \times \mathbb{R}^n_+$, and each $i \in N$,
$$d_i^{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y}, w) = \begin{cases} w & \begin{array}{l} \text{if } x_i = 0,\ \sigma_i > \sigma_j \text{ for each } j \neq i, \\ \text{and } y_i > \max\left\{ f\left(\sum_{j \neq i} \sigma_j \bar{x}\right), w \right\} \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

The function $d^{\boldsymbol{y}}$ assigns all of the output produced to only one agent who provides the maximal *positive* amount, but less than $\bar{x}$, of labor time and reports zero demand for output. The function $d^{\boldsymbol{\sigma}}$ assigns all of the output produced to only one agent who reports the highest skill and does not work.

Let us assume that a game form in Γ requires agents to announce their own skills. Moreover, assume that this game form specifies the share of output by using the function $d^{\boldsymbol{\sigma}}$ whenever the expected output $f\left(\sum \sigma_k x_k\right)$

derived from the deta $(\boldsymbol{\sigma},\boldsymbol{x})$ differs from the realized output $w(\boldsymbol{s},\boldsymbol{x})$. Then, it has an interesting property as the following Lemma A1 shows.

Lemma A1: *Let Assumption 1 hold. Let $\gamma = (A, g) \in \Gamma$ be a game form, where every agent i is requested to announce his/her own skill, σ_i, as a part of a message, and the outcome function g specifies the share of the output according to the function $d^{\boldsymbol{\sigma}}$ whenever $f\left(\sum \sigma_i x_i\right) \neq w(\boldsymbol{s},\boldsymbol{x})$. Given $(\boldsymbol{u},\boldsymbol{s}) \in \mathcal{E}$, let $\boldsymbol{a} = (\boldsymbol{m},\widehat{\boldsymbol{x}}) \in \times_{i\in N} A_i$ be a Nash equilibrium of $(\gamma,\boldsymbol{u},\boldsymbol{s})$ such that $f\left(\sum \sigma_k \widehat{x}_k\right) = w(\boldsymbol{s},\widehat{\boldsymbol{x}})$. Then, for each $i \in N$ with $\widehat{x}_i > 0$, $\sigma_i = s_i$.*

This proof is presented exactly as Lemma 1 in Yamada and Yoshihara (2007). Note that the mechanism, γ^*, constructed in the proof of Theorem 2 meet the premise of Lemma A1.

Proof of Theorem 2.

As a preliminary step, given $\boldsymbol{s} \in \mathcal{S}^n$, $\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \in Z^n$, and $p \in \Delta$, let $B(p, s_i, z_i) \equiv \{z_i' \in Z \mid p_y y_i' - p_x s_i x_i' \leq p_y y_i - p_x s_i x_i\}$. Moreover, define $B\left(\Delta^P(\boldsymbol{e},\boldsymbol{z}), s_i, z_i\right) \equiv \cup_{p\in\Delta^P(\boldsymbol{e},\boldsymbol{z})} B(p, s_i, z_i)$.

Let $p_\alpha(x_i; \boldsymbol{x}_{-i}, \boldsymbol{s}) \equiv \lim_{x_i'\to x_i} \frac{f\left(\sum_{j\neq i} s_j x_j + s_i x_i'\right) - f\left(\sum_{j\neq i} s_j x_j + s_i x_i\right)}{s_i x_i' - s_i x_i} s_i$, where $\alpha =$ '+' if $x_i' > x_i$; and $\alpha =$'−' if $x_i' < x_i$. Given $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y}) \in \mathcal{U}^n \times \mathcal{S}^n \times [0,\bar{x}]^n \times \mathbb{R}^n_+$, let

$$\begin{aligned} & N(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y}) \\ \equiv\ & \left\{ i \in N \mid \exists (x_i^\circ, y_i^\circ) \in Z \text{ s.t. } \left((x_i^\circ, \boldsymbol{x}_{-i}), \left(y_i^\circ, \boldsymbol{y}_{-i}\right)\right) \in \varphi(\boldsymbol{u},\boldsymbol{\sigma}) \right\}. \end{aligned}$$

Let us see how the set $N(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y})$ can function in the mechanism. When $i \in N(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y})$, there are two cases: one is $x_i = 0$, and the other is $x_i > 0$.

Let $x_i = 0$ for $i \in N(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y})$. Then, σ_i may be a false announcement whenever there exists $(\sigma_i^\circ, x_i^\circ, y_i^\circ)$, rather than just (x_i°, y_i°), such that $\left((x_i^\circ, \boldsymbol{x}_{-i}), \left(y_i^\circ, \boldsymbol{y}_{-i}\right)\right) \in \varphi(\boldsymbol{u}, (\sigma_i^\circ, \boldsymbol{\sigma}_{-i}))$ holds. If such a profile exists, then this agent could be a potential deviator in her announcements of not only her consumption vector but also her skill. Therefore, the mechanism would assign a 'punishment outcome' to this agent by taking $(\sigma_i^\circ, x_i^\circ, y_i^\circ)$ as a potential true message. However, such a potential true message would not necessarily be uniquely specified, in that there may be multiple potential true messages. In such a case, we specify how to select one from the set of multiple potential true messages in the following way. For $i \in N(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{x},\boldsymbol{y})$ with $x_i = 0$, let

$(\sigma_i^\mu, x_i^\mu, y_i^\mu)$ be selected by:

$$(\sigma_i^\mu, x_i^\mu, y_i^\mu) = \arg \min_{\left\{\left(\sigma_i^\circ, x_i^\circ, y_i^\circ\right) \mid \left(\left(x_i^\circ, \boldsymbol{x}_{-i}\right), \left(y_i^\circ, \boldsymbol{y}_{-i}\right)\right) \in \varphi\left(\boldsymbol{u}, \left(\sigma_i^\circ, \boldsymbol{\sigma}_{-i}\right)\right)\right\}} y_i^\circ - p_\alpha\left(x_i^\circ; \boldsymbol{x}_{-i}, \left(\sigma_i^\circ, \boldsymbol{\sigma}_{-i}\right)\right) x_i^\circ.$$

Here, if $\sigma_i^\mu x_i^\mu > 0$, then $y_i^\mu - p_\alpha\left(x_i^\mu; \boldsymbol{x}_{-i}, \left(\sigma_i^\mu, \boldsymbol{\sigma}_{-i}\right)\right) x_i^\mu = y_i^\mu - p_-\left(x_i^\mu; \boldsymbol{x}_{-i}, \left(\sigma_i^\mu, \boldsymbol{\sigma}_{-i}\right)\right) x_i^\mu$, while if $\sigma_i^\mu x_i^\mu = 0$, then $y_i^\mu - p_\alpha\left(x_i^\mu; \boldsymbol{x}_{-i}, \left(\sigma_i^\mu, \boldsymbol{\sigma}_{-i}\right)\right) x_i^\mu = y_i^\mu - p_+\left(x_i^\mu; \boldsymbol{x}_{-i}, \left(\sigma_i^\mu, \boldsymbol{\sigma}_{-i}\right)\right) x_i^\mu = y_i^\mu$.

Next, consider $x_i > 0$ for $i \in N(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y})$. In this case, σ_i may be the ture information as discussed below. Then, as there exists (x_i°, y_i°) such that $\left(\left(x_i^\circ, \boldsymbol{x}_{-i}\right), \left(y_i^\circ, \boldsymbol{y}_{-i}\right)\right) \in \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$ by definition of $N(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{x}, \boldsymbol{y})$, such (x_i°, y_i°) is a potential true message for agent i. Then, the mechanism would assign a 'punishment outcome' to this agent by taking (x_i°, y_i°) as a potential true message. Note that, again there may be multiple potential true messages. However, in this case, any selection from the set of potential true messages leaves the agent indifferent, as $u_i(x_i^\circ, y_i^\circ) = u_i(x_i'^\circ, y_i'^\circ)$ holds for any different potential true messages (x_i°, y_i°) and $(x_i'^\circ, y_i'^\circ)$, given that $\varphi(\boldsymbol{u}, \boldsymbol{\sigma}) \subseteq P(\boldsymbol{u}, \boldsymbol{\sigma})$. Therefore, in this case, the specification of a 'punishment outcome' is not difficult, as discussed below.

Now, we are ready to discuss the construction of a mechanism. Denote the upper boundary of $L(z_i, u_i)$ by $\partial L(z_i, u_i) \equiv \{z_i' \in L(z_i, u_i) \mid u_i(z_i') = u_i(z_i)\}$. For the sake of simplifying notations, given $(\boldsymbol{u}, \boldsymbol{\sigma}) \in \mathcal{E}$ and $\left(z_j^\circ, \left(\boldsymbol{x}_{-j}, \boldsymbol{y}_{-j}\right)\right) \in \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$, we will sometimes use $L\left(z_j^\circ, u_j; \boldsymbol{\sigma}\right)$ instead of the precise statement $L\left(\left(z_j^\circ, \left(\boldsymbol{x}_{-j}, \boldsymbol{y}_{-j}\right)\right), u_i; \boldsymbol{\sigma}\right)$ in the following discussion. We define a canonical mechanism $\gamma^* = (A^*, g^*) \in \Gamma_C^*$ with $A_i^* \equiv M_i \times [0, \bar{x}]$, where $M_i \equiv \mathcal{U}^n \times \mathcal{S} \times Z$ with generic element $(\boldsymbol{u}^i, \sigma_i, z_i)$, for each $i \in N$, as follows:

For each $\boldsymbol{a} = \left((\boldsymbol{u}^i)_{i\in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \widehat{\boldsymbol{x}}\right) \equiv (\boldsymbol{u}^i, \sigma_i, z_i, \widehat{x}_i)_{i\in N} \in \times_{i\in N}(M_i \times [0, \bar{x}])$ and for each output $w = w(\boldsymbol{s}, \widehat{\boldsymbol{x}}) \in \mathbb{R}_+$,

Rule 1: if $f\left(\sum \sigma_k \widehat{x}_k\right) = w$, and

1-1: there exists $\boldsymbol{u} \in \mathcal{U}^n$ such that $\boldsymbol{u}^i = \boldsymbol{u}$ for each $i \in N$, and $\sum y_k \leq w$ and

1-1-a): if $(\widehat{\boldsymbol{x}}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$, then $g^*(\boldsymbol{a}; w) = (\widehat{\boldsymbol{x}}, \boldsymbol{y})$,

1-1-b): if $(\widehat{\boldsymbol{x}}, \boldsymbol{y}) \notin \varphi(\boldsymbol{u}, \boldsymbol{\sigma})$, then $g^*(\boldsymbol{a}; w) = (\widehat{\boldsymbol{x}}, \boldsymbol{0})$,

1-2: there exists $j \in N$ such that $\boldsymbol{u}^i = \boldsymbol{u} \neq \boldsymbol{u}^j$ for each $i \neq j$, $(\widehat{\boldsymbol{x}}, \boldsymbol{y}) \notin \varphi(\boldsymbol{u}^j, \boldsymbol{\sigma})$, and

1-2-a): if $j \in N(\boldsymbol{u}, \boldsymbol{\sigma}, \widehat{\boldsymbol{x}}, \boldsymbol{y})$, then $g_i^*(\boldsymbol{a}; w) = (\widehat{x}_i, 0)$ for each $i \neq j$, and

$$g_j^*(\boldsymbol{a}; w) = \begin{cases} \left(\widehat{x}_j, \min\left\{y_j'', w\right\}\right) & \text{if } y_j > f\left(\sum \sigma_k \bar{x}\right) \\ \left(\widehat{x}_j, 0\right) & \text{otherwise,} \end{cases}$$

where y_j'' is given by[19]

$$\left(\widehat{x}_j, y_j''\right) \in \begin{cases} \partial\left[L\left(z_j^\circ, u_j; \boldsymbol{\sigma}\right) \cup B\left(\Delta^P\left(\left(\boldsymbol{u}, \boldsymbol{\sigma}\right), \left(z_j^\circ, \left(\widehat{\boldsymbol{x}}_{-j}, \boldsymbol{y}_{-j}\right)\right)\right), \sigma_j, z_j^\circ\right)\right] & \text{if } \widehat{x}_j > 0 \\ \left\{\left(0, y_j^\mu - p_\alpha\left(x_j^\mu; \widehat{\boldsymbol{x}}_{-j}, \left(\sigma_j^\mu, \boldsymbol{\sigma}_{-j}\right)\right) x_j^\mu\right)\right\} & \text{otherwise,} \end{cases}$$

for $z_j^\circ = (x_j^\circ, y_j^\circ)$ with $\left(\left(x_j^\circ, \widehat{\boldsymbol{x}}_{-j}\right), \left(y_j^\circ, \boldsymbol{y}_{-j}\right)\right) \in \varphi\left(\boldsymbol{u}, \boldsymbol{\sigma}\right)$,

1-2-b): if $j \notin N\left(\boldsymbol{u}, \boldsymbol{\sigma}, \widehat{\boldsymbol{x}}, \boldsymbol{y}\right)$ and there exists $(x_j^\circ, y_j^\circ) \in Z$ such that $\left(\left(x_j^\circ, \widehat{\boldsymbol{x}}_{-j}\right), \left(y_j^\circ, \boldsymbol{y}_{-j}\right)\right) \in P\left(\boldsymbol{u}^j, \boldsymbol{\sigma}\right)$, then $g_i^*\left(\boldsymbol{a}; w\right) = (\widehat{x}_i, 0)$ for each $i \neq j$, and

$$g_j^*\left(\boldsymbol{a}; w\right) = \begin{cases} (\widehat{x}_j, w) & \text{if } \widehat{x}_j > 0 \text{ and } y_j > w \\ (\widehat{x}_j, 0) & \text{otherwise,} \end{cases}$$

1-3: in any other case, $g^*\left(\boldsymbol{a}; w\right) = \left(\boldsymbol{x}, d^{\boldsymbol{y}}\left(\boldsymbol{\sigma}, \widehat{\boldsymbol{x}}, \boldsymbol{y}, w\right)\right)$,

Rule 2: if $f\left(\sum \sigma_k \widehat{x}_k\right) \neq w$, then $g^*\left(\boldsymbol{a}; w\right) = \left(\boldsymbol{x}, d^{\boldsymbol{\sigma}}\left(\boldsymbol{\sigma}, \widehat{\boldsymbol{x}}, \boldsymbol{y}, w\right)\right)$.

Lemma A2: *Let Assumption 1 hold and* $n \geq 3$. *Then,* γ^* implements *any interior and efficient SCC* φ *satisfying* **M** *and* **NUS** in Nash equilibria.

Proof. Let φ be an interior and efficient *SCC* satisfying **M** and **NUS**. Let $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{E}$.

(1) First, we show that $\varphi\left(\boldsymbol{e}\right) \subseteq NA\left(\gamma^*, \boldsymbol{e}\right)$. Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in \varphi\left(\boldsymbol{e}\right)$. Let $\boldsymbol{a} = \left(\left(\boldsymbol{u}^i\right)_{i \in N}, \boldsymbol{s}, \boldsymbol{z}, \boldsymbol{x}\right) \in \times_{i \in N}\left(M_i \times [0, \bar{x}]\right)$ be such that $\boldsymbol{u}^i = \boldsymbol{u}$ for each $i \in N$. Then, $g^*\left(\boldsymbol{a}\right) = (\boldsymbol{x}, \boldsymbol{y})$ from Rule 1-1.[20] Suppose $j \in N$ deviates to $a_j' = \left(\boldsymbol{u}^{j\prime}, s_j', z_j', x_j'\right) \in M_i \times [0, \bar{x}]$. From Assumption 1 and the continuity of the utility functions, if $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = 0$, it implies the worst outcome for j.

If a_j' induces Rule 2, then $x_j' > 0$ and $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = 0$. If a_j' induces Rule 1-3, then either $\left(\left(x_j', \boldsymbol{x}_{-j}\right), \left(y_j', \boldsymbol{y}_{-j}\right)\right) \in \varphi\left(\boldsymbol{u}^j, \left(s_j', \boldsymbol{s}_{-j}\right)\right)$ or $\sum_{i \neq j} y_i + y_j' > f\left(\sum_{i \neq j} s_i x_i + s_j x_j'\right)$. In either case, $y_j' > 0$ holds, and so $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = 0$. If a_j' induces Rule 1-2-b, then $x_j = 0 = x_j'$ and $s_j' \neq s_j$. Thus, $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = 0$.

If a_j' induces Rule 1-2-a, then either $x_j' > 0$ or $x_j' = 0$. In the former case, where $s_j' = s_j$ holds, Rule 1-2-a implies that

$$g_j^*\left(a_j', \boldsymbol{a}_{-j}\right) \in \partial\left[L\left(z_j, u_j; \boldsymbol{s}\right) \cup B\left(\Delta^P\left(\boldsymbol{e}, \boldsymbol{z}\right), s_j, z_j\right)\right] \subseteq L\left(z_j, u_j\right),$$

since any price vector $p \in \Delta^P\left(\boldsymbol{e}, \boldsymbol{z}\right)$ implies $B\left(p, s_j, z_j\right) \subseteq L\left(z_j, u_j\right)$. When $x_j' = 0$, there exists $\left(\sigma_j^\mu, x_j^\mu, y_j^\mu\right)$ such that $\left(\left(x_j^\mu, \boldsymbol{x}_{-j}\right), \left(y_j^\mu, \boldsymbol{y}_{-j}\right)\right) \in \varphi\left(\boldsymbol{u}, \left(\sigma_j^\mu, \boldsymbol{s}_{-j}\right)\right)$

[19] Note ∂X denotes the upper boundary of the set $X \subseteq \mathbb{R}_+^2$.

[20] From now on, we simply write a value of g^* as $g^*\left(\boldsymbol{a}\right)$ instead of $g^*\left(\boldsymbol{a}; f\left(\sum s_k x_k\right)\right)$, without loss of generality. Moreover, let $g_{j2}^*\left(\boldsymbol{a}\right)$ be the second component of $g_j^*\left(\boldsymbol{a}\right)$, which is the share of the output produced to agent j specified by the mechanism under $\boldsymbol{a}$.

and $y_j^\mu - p_-\left(x_j^\mu; \boldsymbol{x}_{-j}, \left(\sigma_j^\mu, \boldsymbol{s}_{-j}\right)\right) x_j^\mu \leq y_j - p_-\left(x_j; \boldsymbol{x}_{-j}, \boldsymbol{s}\right) x_j$ or $y_j^\mu \leq y_j - p_-\left(x_j; \boldsymbol{x}_{-j}, \boldsymbol{s}\right) x_j$ hold. Let $p = (p_x, p_y)$ be the efficiency price which supports $\boldsymbol{z}$ as a φ-optimal allocation at $\boldsymbol{e}$. Then, $y_j - \frac{p_x}{p_y} s_j x_j \geq y_j - p_-\left(x_j; \boldsymbol{x}_{-j}, \boldsymbol{s}\right) x_j$, so that $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = y_j^\mu - p_\alpha\left(x_j^\mu; \boldsymbol{x}_{-j}, \left(\sigma_j^\mu, \boldsymbol{s}_{-j}\right)\right) x_j^\mu \leq y_j - \frac{p_x}{p_y} s_j x_j$. This implies $g_j^*\left(a_j', \boldsymbol{a}_{-j}\right) \in L\left(z_j, u_j\right)$. Finally, if a_j' induces Rule 1-1, then $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = y_j' \leq f(\sum_{i \neq j} s_i x_i + s_j x_j') - \sum_{i \neq j} y_i$. Thus, since $\boldsymbol{z} \in P\left(\boldsymbol{e}\right)$, $u_j\left(x_j', y_j'\right) \leq u_j\left(z_j\right)$. In summary, j has no incentive to switch to a_j'.

(2) Second, show $NA\left(\gamma^*, \boldsymbol{e}\right) \subseteq \varphi\left(\boldsymbol{e}\right)$. Since g^* is constant with $\boldsymbol{x}$ in $\boldsymbol{z}$, let $\boldsymbol{a} = \left(\left(\boldsymbol{v}^i\right)_{i \in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \boldsymbol{x}\right) \in NE\left(\gamma^*, \boldsymbol{e}\right)$ without loss of generality.

Suppose that $\boldsymbol{a}$ induces Rule 2. Then, either $N^0\left(\boldsymbol{x}\right) \equiv \{i \in N \mid x_i = 0\} = \varnothing$ or $N^0\left(\boldsymbol{x}\right) \neq \varnothing$. If $N^0\left(\boldsymbol{x}\right) = \varnothing$, then for each $i \in N$, $g_{i2}^*\left(\boldsymbol{a}\right) = 0$. Then, if $\sum_{i \neq k} \sigma_i x_i = \sum_{i \neq k} s_i x_i$ holds for each $k \in N$, then $(n-1) \cdot \left(\sum \sigma_i x_i\right) = (n-1) \cdot \left(\sum s_i x_i\right)$, which contradicts Rule 2. Thus, for some $j \in N$, $\sum_{i \neq j} \sigma_i x_i \neq \sum_{i \neq j} s_i x_i$. If j switches to $a_j' = \left(\boldsymbol{v}^{j\prime}, \sigma_j', z_j', x_j'\right)$ with $\sigma_j' > \max\left\{\sigma_i \mid i \neq j\right\}$, $y_j' > \max\left\{f\left(\sum_{k \neq j} \sigma_k \overline{x}\right), w\right\}$, and $x_j' = 0$, then $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) > 0$ under Rule 2.

Let $N^0\left(\boldsymbol{x}\right) \neq \varnothing$ with $\# N^0\left(\boldsymbol{x}\right) \geq 2$. Then, for each $j \in N^0\left(\boldsymbol{x}\right)$, if j's deviating strategy a_j' is such that for each $i \neq j$, $\sigma_j' > \sigma_i$, $y_j' > \max\left\{f\left(\sum_{k \neq j} \sigma_k \overline{x}\right), w\right\}$, and $x_j' = 0$, then $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) = f\left(\sum s_k x_k\right)$ under Rule 2.

Let $\# N^0\left(\boldsymbol{x}\right) = 1$ and $\# N \backslash N^0\left(\boldsymbol{x}\right) \geq 2$. Then, there exists $j \in N \backslash N^0\left(\boldsymbol{x}\right)$ such that $\sum_{i \in N \backslash (N^0(\boldsymbol{x}) \cup \{j\})} \sigma_i x_i \neq \sum_{i \in N \backslash (N^0(\boldsymbol{x}) \cup \{j\})} s_i x_i$. Thus, j can switch to a_j' such that $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) > 0$ under Rule 2. This can be shown in a similar way to the case of $N^0\left(\boldsymbol{x}\right) = \varnothing$.

Suppose that $\boldsymbol{a}$ induces Rules 1-2 or 1-3. Then, there exists $j \in N$ such that $g_{j2}^*\left(\boldsymbol{a}\right) = 0$. From Lemma A1, $\sigma_j = s_j$ or $x_j = 0$. Suppose $\boldsymbol{a}$ induces Rule 1-2. Then, $g_{j2}^*\left(\boldsymbol{a}\right) = 0$ implies that $y_j \leq f\left(\sum \sigma_k \overline{x}\right)$. Then, j can either deviate to Rule 1-3 with $\sigma_j' = s_j$, $x_j' = \pi\left(\boldsymbol{x}_{-j}\right) < \overline{x}$, and $y_j' = 0$ or get $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) > 0$ under Rule 1-2 by $y_j' > f\left(\sum_{i \neq j} \sigma_i \bar{x} + \sigma_j' \bar{x}\right)$. Suppose $\boldsymbol{a}$ induces Rule 1-3. Then, there exists a_j' such that $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) > 0$ under Rule 1-3. In summary, $\boldsymbol{a}$ induces neither Rule 1-2 nor Rule 1-3.

Suppose that $\boldsymbol{a}$ induces Rule 1-1-b. Then, $g^*\left(\boldsymbol{a}\right) = \left(\boldsymbol{x}, \boldsymbol{0}\right)$. Then, some $j \in N$ can deviate to induce Rule 1-2, so that $g_{j2}^*\left(a_j', \boldsymbol{a}_{-j}\right) > 0$, which is a contradiction.

Thus, $\boldsymbol{a}$ induces Rule 1-1-a, and $g^*(\boldsymbol{a}) = (\boldsymbol{x}, \boldsymbol{y})$. From the definition of Rule 1-1-a, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}', \boldsymbol{\sigma})$ where $\boldsymbol{u}' = \boldsymbol{v}^i$ for all $i \in N$. Since $\boldsymbol{a} \in NE(\gamma^*, \boldsymbol{e})$, $\sigma_i = s_i$ holds for any $i \in N$ with $x_i > 0$ according to Lemma A1. Assume, without loss of generality, that there exists at most one unique individual j such that $x_j = 0$. Let us consider the following two cases:

Case 1: Let $(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{u}', \boldsymbol{s})$. Then, we can show that $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}', \boldsymbol{s})$. Suppose that $(\boldsymbol{x}, \boldsymbol{y}) \notin \varphi(\boldsymbol{u}', \boldsymbol{s})$. Then, for the individual $j \in N$ with $x_j = 0$, $\sigma_j \neq s_j$. Then, if this j takes the strategy $a'_j = (\boldsymbol{u}^j, s_j, z'_j, x'_j)$ with $\boldsymbol{u}^j \neq \boldsymbol{u}'$, $x'_j > 0$, and $y'_j > f(\sum s_k \overline{x})$, then Rule 1-2-b can be applied. This is because **NUS** implies $j \notin N(\boldsymbol{u}', \boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$ by $(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{u}', \boldsymbol{s}) \backslash \varphi(\boldsymbol{u}', \boldsymbol{s})$ and $x_j = 0$. Then, if $x'_j > 0$ is sufficiently small, $u_j(g^*_j(a'_j, \boldsymbol{a}_{-j})) = u_j\left(x'_j, f\left(\sum_{i \neq j} s_i x_i + s_j x'_j\right)\right) > u_j(0, y_j) = u_j(g^*_j(\boldsymbol{a}))$ holds by the fact that $(\boldsymbol{x}, \boldsymbol{y})$ is an interior allocation, since $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}', \boldsymbol{\sigma})$. This implies $(\boldsymbol{x}, \boldsymbol{y}) \notin NA(\gamma^*, (\boldsymbol{u}, \boldsymbol{s}))$, which is a contradiction. Thus, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}', \boldsymbol{s})$. Note that $(\boldsymbol{x}, \boldsymbol{y}) \in NA(\gamma^*, (\boldsymbol{u}, \boldsymbol{s}))$ implies that $\partial L((x_i, y_i), u'_i; \boldsymbol{s}) \subseteq L((x_i, y_i), u_i) \cap Z_i(\boldsymbol{s}, \boldsymbol{z}_{-i})$ for each $i \in N$ by Rule 1-2-a. Thus, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{u}, \boldsymbol{s})$ by **M**.

Case 2: Let $(\boldsymbol{x}, \boldsymbol{y}) \notin P(\boldsymbol{u}', \boldsymbol{s})$. Since $(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{u}', \boldsymbol{\sigma})$, $\sigma_j < s_j$ holds for the agent $j \in N$ with $x_j = 0$. In this case, either $j \in N(\boldsymbol{u}', \boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$ or not.

First, suppose that $j \notin N(\boldsymbol{u}', \boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$. Take $\boldsymbol{u}^j = (u''_j, \boldsymbol{u}'_{-j})$ such that

$$\partial\left[L\left((x_j, y_j), u'_j; \boldsymbol{s}\right) \cup \left(\cup_{p \in \Delta^P((\boldsymbol{u}', \boldsymbol{\sigma}), (\boldsymbol{x}, \boldsymbol{y}))} B(p, s_j, (x_j, y_j))\right)\right] \subseteq L\left((x_j, y_j), u''_j\right).$$

Then, $(\boldsymbol{x}, \boldsymbol{y}) \in P(\boldsymbol{u}^j, \boldsymbol{s})$ holds. Then, by $a'_j = (\boldsymbol{u}^j, s_j, z^*_j, x^*_j)$ with $x^*_j > 0$ and $y^*_j > f(\sum s_k \overline{x})$, j can induce Rule 1-2-b, and get $g^*_{j2}(a'_j, \boldsymbol{a}_{-j}) = f\left(\sum_{i \neq j} s_i x_i + s_j x^*_j\right)$. Thus, if $x^*_j > 0$ is sufficiently small, $(\boldsymbol{x}, \boldsymbol{y}) \notin NA(\gamma^{**}, (\boldsymbol{u}, \boldsymbol{s}))$, which is a contradiction. Thus, $j \notin N(\boldsymbol{u}', \boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$ does not hold.

Second, let $j \in N(\boldsymbol{u}', \boldsymbol{s}, \boldsymbol{x}, \boldsymbol{y})$. Since $(\boldsymbol{x}, \boldsymbol{y}) \notin P(\boldsymbol{u}', \boldsymbol{s})$, there exists $z'_j \equiv (x'_j, y'_j) \gg \boldsymbol{0}$ such that $(z'_j, (\boldsymbol{x}_{-j}, \boldsymbol{y}_{-j})) \in \varphi(\boldsymbol{u}', \boldsymbol{s})$. In this case, $y'_j \geq y_j + \frac{p_x}{p_y} s_j x'_j$ holds for $p \equiv (p_x, p_y) \in \Delta^P((\boldsymbol{u}', \boldsymbol{s}), (z'_j, (\boldsymbol{x}_{-j}, \boldsymbol{y}_{-j})))$, since $z_j, z'_j \in \left\{(x, y) \in Z \mid y = f(\sum_{i \neq j} s_i x_i + s_j x) - \sum_{i \neq j} y_i\right\}$ and f is concave. Moreover, since $z'_j \notin L((x_j, y_j), u'_j)$ and $z'_j \in B(p, s_j, z'_j)$, $B(p, s_j, z'_j) \nsubseteq L((x_j, y_j), u'_j; \boldsymbol{s})$ holds. Then, since $\boldsymbol{a} \in NE(\gamma^*, \boldsymbol{e})$, (i) $\partial\left[L((x_j, y_j), u'_j; \boldsymbol{s}) \cup B(p, s_j, z'_j)\right] \subseteq L((x_j, y_j), u_j)$ and (ii) $y'_j = y_j + \frac{p_x}{p_y} s_j x'_j$ hold. Indeed, if (i) does not hold, then j can induce Rule 1-2-a by suitably choosing $a'_j = (\boldsymbol{u}^j, s_j, z^*_j, x^*_j)$ with $\boldsymbol{u}^j \neq \boldsymbol{u}'$, $x^*_j > 0$, and $y^*_j > f(\sum s_k \overline{x})$, so that $g^*_{j2}(a'_j, \boldsymbol{a}_{-j}) \geq y_j + \frac{p_x}{p_y} s_j x^*_j$ and

$g_j^*\left(a_j', \boldsymbol{a}_{-j}\right) \notin L\left(\left(x_j, y_j\right), u_j\right)$, which is a contradiction from $\boldsymbol{a} \in NE\left(\gamma^{**}, \boldsymbol{e}\right)$. The same argument also follows if (ii) does not hold.

Consider $\boldsymbol{u}'' \equiv \left(u_j, \boldsymbol{u}_{-j}'\right)$. Note that by **M**, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi\left(\boldsymbol{u}'', \boldsymbol{\sigma}\right)$. Then, by Rule 1-1-a, for $\boldsymbol{a}'' = \left(\left(\boldsymbol{v}''^i\right)_{i \in N}, \boldsymbol{\sigma}, \boldsymbol{z}, \boldsymbol{x}\right)$ with $\boldsymbol{v}''^i = \boldsymbol{u}''$ $(\forall i \in N)$, $g^*\left(\boldsymbol{a}''\right) = (\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{a}'' \in NE\left(\gamma^*, \boldsymbol{e}\right)$ hold. Suppose $(\boldsymbol{x}, \boldsymbol{y}) \notin \varphi\left(\boldsymbol{u}'', \boldsymbol{s}\right)$. Then, since $(\boldsymbol{x}, \boldsymbol{y}) \in P\left(\boldsymbol{u}'', \boldsymbol{s}\right)$, j can induce Rule 1-2-b by the same reasoning as in Case 1, so that $(\boldsymbol{x}, \boldsymbol{y}) \notin NA\left(\gamma^*, \boldsymbol{e}\right)$, which is a contradiction. Thus, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi\left(\boldsymbol{u}'', \boldsymbol{s}\right)$. Then, $\boldsymbol{a}'' \in NE\left(\gamma^*, \boldsymbol{e}\right)$ and $(\boldsymbol{x}, \boldsymbol{y}) \in NA\left(\gamma^*, \boldsymbol{e}\right)$ imply that $\partial L\left(\left(x_i, y_i\right), u_i'; \boldsymbol{s}\right) \subseteq L\left(\left(x_i, y_i\right), u_i\right)$ holds for each $i \in N \setminus \{j\}$ by Rule 1-2-a. Thus, $(\boldsymbol{x}, \boldsymbol{y}) \in \varphi(\boldsymbol{e})$ by **M**. ■

Proof of Theorem 2. From **Lemma A2**, we obtain the desired result. ■

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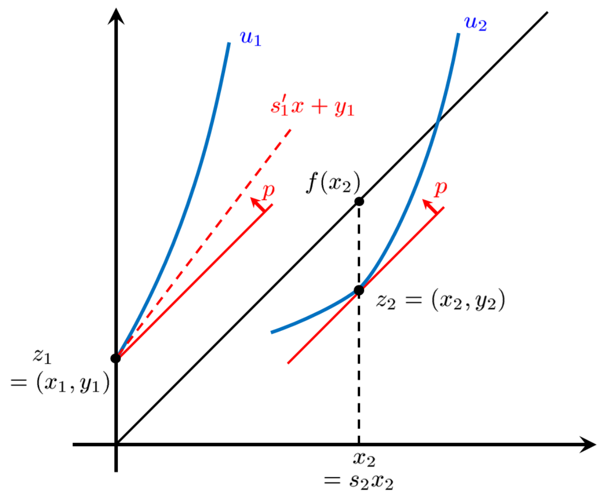
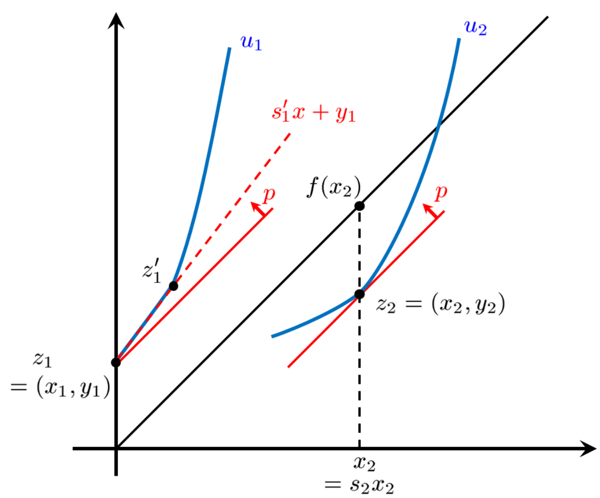


$s_1 = 1 < s_1'$, $s_2 = 1$, $(z_1, z_2) \in \varphi(e)$ where $e = ((u_1, u_2), (s_1, s_2))$.
Moreover, for $e' = ((u_1, u_2), (s_1', s_2))$, $(z_1, z_2) \in P(e') \setminus \varphi(e')$.

φ satisfies NUS because there is no $z_1' \in Z$ such that $(z_1', z_2) \in \varphi(e')$

Figure 1a



$s_1 = 1 < s_1'$, $s_2 = 1$, $(z_1, z_2) \in \varphi(e)$ where $e = ((u_1, u_2), (s_1, s_2))$.
Moreover, for $e' = ((u_1, u_2), (s_1', s_2))$, $(z_1, z_2) \in P(e') \setminus \varphi(e')$.

φ may not satisfy NUS because $(z_1', z_2) \in P(e')$ may be φ-optimal at e'

Figure 1b