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On the Existence and Characterization of Unequal Exchange in the Free Trade Equilibrium*

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Abstract

As in Roemer (1982, chapter 1), this paper considers a simple international trade model and examines the existence and characterization of free trade equilibria involving the unequal exchange of labor (UE). The paper provides an almost complete characterization of the domain of economies in which free trade equilibria with incomplete specialization exist. Moreover, the necessary and sufficient conditions for free trade equilibrium to involve UE is identified. It suggests that the emergence of free trade equilibria with UE cannot be entailed by the competitive mechanism of markets and unequal distribution of wealth alone, but might be understood as an outcome of equilibrium selection on the basis of Nash bargaining between rich and poor nations.

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1 Introduction

Understanding why some countries in the world economy are so rich while some are so poor is one of the most important issues in economics, as there are large inequalities in income per capita and output per worker across countries, increasing since 1820.[1] Regarding this issue, the so-called *dependence school* recognizes the emergence of development and underdevelopment in the capitalistic world system as a product of *exploitative relations between rich and poor nations*. For instance, among others, Emmanuel (1972) discusses the generation of *unequal exchange* (UE) between rich and poor nations due to the *core-periphery structure* of international economies.[2] He argued, that in the world economy characterized by customary disparity in wage rates among developed and undeveloped nations, the international trade of commodities and capital mobility across nations cause the transfer of surplus labor from poor nations with lower capital-labor rations to wealthy nations with higher capital-labor ratios, which results in the impoverishment of poor nations and the enrichment of wealthy ones.

Samuelson (1976) argues that Emmanuel's theory of UE is inconsistent with the theory of comparative advantage, which implies the existence of mutual gain from trade. However, at best, his criticism refutes the second claim of the theory and not the first, which illustrates a mechanism for generating UE. Indeed, the generation of UE might be compatible with mutual gains from trade; as Marx (1968, chapter 20, (e)) notes, "a richer country exploits a poorer one, even when the latter benefits from the exchange." Marx's observation suggests that imperfectly competitive exchange conditions, such as an institutionalized wage disparity, are not an essential source of UE, though they may exacerbate the UE feature of international economies.

Roemer (1982, chapter 1; 1983) provides a coherent formal analysis of the generation of UE using simple models of perfectly competitive markets. He

[1] For instance, see Acemoglu (2009, Chapter 1).

[2] The question of unequal exchange in international trade has been historically argued, as Marx (1954, chapter 20) explains exploitation among nations as a consequence of the disparity in wage rates, due to differences in the technologies available to them.

considers free trade equilibria in international economies where all nations have identical technology and labor supply but unequal capital-labor ratios, and there are international markets for commodities but not capital nor labor. His main purpose is to exhibit in as simple a framework as possible, that inequality of capital endowments among nations and competitive markets are sufficient institutions for an exploitative UE, and the existence of UE is compatible with the presence of mutual gains from trade.

However, there remain many unexamined issues in characterizing the essential mechanism giving rise to UE in competitive international economies. Indeed, though in Roemer (1982, chapter 1), the existence of UE is discussed in the simplest model of an economy with labor-minimizing agents, it is argued by only providing a numerical example. In Roemer (1983, Theorem 1), a simple model of international economies is given in which each nation's objective is to maximize the monetary value of its own capital, and only equilibria featuring *complete* specialization are analyzed.[3]

In this paper, sharing the same perspective as Marx (1968, chapter 20, (e)) and Roemer (1982, chapter 1; 1983), we consider a simple model of competitive international trade in which all nations have the same technology and population size as well as domestic labor and capital markets, but only commodity markets are international in scope.[4] On this basis, we provide a more comprehensive analysis of the conditions giving rise to UE in international economies. Our model follows Roemer's in assuming Leontief technology, with no choice among techniques or technical change, and common welfare functions, our goal being to show how UE might arise even without differences in preferences for leisure and commodity consumption.[5]

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[3]That is, each nation is specialized toward production activities at a *proper subset* of all sectors in equilibrium; therefore, each nation's domestic wage rate and interest rate differ. In such an equilibrium, it is shown that the nation with zero wage rate is UE exploited, and the nation with zero interest rate is UE exploiting.

[4]This feature is also shared with the basic model of the standard international trade theory a la Heckscher-Ohlin-Samuelson. This setting is reasonable, at least for discussing a universal long-run feature of free trade equilibria, since any (advanced) knowledge of technology can be dispersed across nations and becomes common in the long run.

[5]Assuming the common welfare function of all nations is consistent with a shared view on exploitation, that an unequal exchange transfer of products due only to differences in preferences for income and leisure is not unjustly exploitative, as Cohen (1995) argues. This is also consistent with the standard approach of international trade theory, in that it argues the generation of international division of labor and the mutual gain from trading, even without assuming differential preferences among traders.

The welfare function presumed in this paper implies that, as in Roemer (1982, chapter 1), all nations are primarily concerned about their citizens' enjoyment of free hours (or leisure time), given that a common subsistence consumption bundle necessary for the citizens' survival is ensured. This model, which we hereafter call a *subsistence international economy*, allows us to analyze competitive international exchange outcomes with the simplification that consumption bundles are insensitive to relative price changes in commodity prices (all nations' citizens always consume the subsistence commodity bundle only)[6], but competitive exchange of commodities will still emerge among nations, due to the division of labor in their production activities. Moreover, whether consumption demand is sensitive to price changes is not essential for the main purpose of this paper. The presumption of subsistence international economies also allows us to examine the generation of UE *independently of the complicated issues involved by capital accumulation*, since as shown later in Section 2, any equilibrium in such economies is characterized by a stationary path with no capital accumulation. To discuss the generation of UE even though the capitalistic motivation of capital accumulation is lacking, it is opportune to presume such economies.

While this paper's model is thus identical to those of Roemer (1982, chapter 1; 1983) in certain core respect, it considers some important extensions of his framework. Firstly, unlike Roemer (1982, chapter 1; 1983), we will focus on an equilibrium where every nation engages in activities of all production sectors (*equilibrium with incomplete specialization*, hereafter), and provide an almost complete characterization for the existence of that equilibrium involving UE. Since here, all nations can access the common Leontief production technique, unlike the standard Ricardian model of international trade, incomplete specialization would be a generic feature of free trade equilibrium. Then, as Theorem 1 of this paper shows, factor price equalization emerges in equilibria with incomplete specialization, even though each nation's prices of labor and capital are determined via its domestic factor markets. Since neither Marx (1954), Emmanuel (1972), nor Roemer (1982, chapter 1; 1983) focuses on such an equilibrium for the subject of UE-generation, the main results in this paper offer a new perspective on conditions giving rise to UE in international trade.

[6]It does not imply that the aggregate demand for each commodity is insensitive to any price change in such economies, since the aggregate demand consists of not only the aggregate consumption demand but also the aggregate demand for that commodity used as a capital good for the next period of production.

Indeed, the first main result characterizes the domain of subsistence economies where free trade equilibrium with incomplete specialization exists. The second main result shows that for any subsistence economy, free trade equilibrium with incomplete specialization involves UE if and only if the initial endowments of financial capital among nations are unequal and the equilibrium prices of commodities are not *labor-value pricing* (See Corollary 1). To see the implications of these results, subsistence economies should be classified into two types. The first type refers to subsistence economies with excessive social endowments of capital stocks. In this case, every incompletely specialized equilibrium involves labor-value pricing and hence no UE, regardless of whether the distribution of financial capital is unequal or not (See Theorem 4).

The second type refers to subsistence economies with non-excessive social endowments of capital stocks. In this case, we may say that such an economy entails *essential technical differences among sectors* if and only if sectoral capital-labor ratios are not equalized in some equilibrium of that economy. Furthermore, the latter condition holds if and only if the unique Frobenius eigenvector of the Leontief matrix of material input coefficients and the vector of the labor input coefficients are linearly independent (See Lemma 1). Since the unique Frobenius eigenvector of the Leontief matrix and the vector of the labor input coefficients are data regarding the economy's production technology, we can check if each such economy entails essential technical differences among sectors, prior to any equilibrium analysis. Then, if an economy with non-excessive capital stocks also features no essential technical difference among sectors, no equilibrium characterized by UE exists (See Theorem 2).

Finally, if an economy with non-excessive capital stocks entails essential technical differences among sectors, we can identify a large domain of initial endowments of financial capital in which the existence of free trade equilibria with incomplete specialization is ensured. Moreover, each such equilibrium involves UE if and only if financial capital is unequally distributed and the corresponding equilibrium price vector entails a positive interest rate (See Theorem 3). Note that an equilibrium price vector with a positive interest rate in such an economy is not labor-value pricing.

Thus, the main results suggest that the existence of UE-Equilibria is secured only in economies with non-excessive capital stocks and essential technical differences among sectors. Even within the class of such economies, however, the inequality of capital endowments among nations and competitive markets alone are insufficient for the realization of an UE-Equilibrium, unlike the main message of Roemer (1982, Chapter 1). This is because free trade equilibria are generically indeterminate in such economies, and so the two institutions alone cannot rule out the realizability of the equilibrium with labor-value pricing. Indeed, in such an economy, the international division of labor occurs even under the equilibrium with labor-value pricing: a wealthier nation is specialized in more capital-intensive production activity while a poorer one in a more labor-intensive activity, through their own optimizing behavior. However, it does not involve the lower labor supply of the former than that of the latter, whenever the equilibrium is labor-value pricing.

Given such results, we provide a counterfactual scenario to realize UE in free trade equilibria. A counterfactual bargaining among nations is considered on the selection of an equilibrium price vector from the set of equilibria. In such bargaining, we observe the existence of an asymmetric power structure among nations due to the unequal distribution of financial capital, which would play a crucial role in eliminating the realization of equilibrium with labor-value pricing. As discussed later, this additional institution would also give us a philosophical explanation as to why international trade characterized by UE might reasonably be understood as exploitative.

The remainder of the paper is organized as follows: Section 2 presents the basic model and an equilibrium notion. Section 3 defines the formulation of exploitation as UE. Section 4 discusses the existence and characterization of free trade equilibria with and without UE. Finally, Section 5 concludes the paper.

2 A Basic Model

Let $\mathcal{N}$ be the set of agents (nations), with cardinality N; there are $n \geqq 2$ commodities. An economy comprises a set of agents $\mathcal{N} = \{1, .., N\}$, with a generic element $\nu \in \mathcal{N}$, and n types of (purely private) commodities are transferable in the market. The production technology, commonly accessible by any agent, is represented by a Leontief production technique (A, L), where A is an $n \times n$ non-negative square matrix of the material input coefficients and L is a $1 \times n$ positive vector of the labor input coefficients. Here, A is assumed to be productive and indecomposable.[7] For the sake of simplicity, let us

[7] Let K be the index set of A's dimension. Then, A is said to be *decomposable* if there is a pair of I and J such that $K = I \cup J$, $I \cap J = \varnothing$, $I, J \neq \varnothing$, and $a_{ij} = 0$ for $i \in I, j \in J$.

assume that for each production period, the maximal amount of labor supply by every agent is equal to unity, and there is no differences in labor skills (human capital) among agents. Let $b \in \mathbb{R}^n_{++}$ be the *subsistence consumption bundle* that every citizen in every nation must consume for his/her survival in one period of production, regardless of whether he/she supplies labor. For simplicity, each nation also has the same population size, normalized to unity. Let $\overline{\omega} \in \mathbb{R}^n_{++}$ be the world endowments of material capital goods at the beginning of the initial period of production.

Assume $\overline{\omega} \geqq A[I-A]^{-1}(Nb)$ and $L[I-A]^{-1}(Nb) < N$.[8] Note that $A[I-A]^{-1}(Nb)$ represents the minimal level of capital stocks necessary for the survival of the economy. Both assumptions imply that the production of aggregate amount of subsistence consumption bundles is technologically feasible in this economy. Moreover, assume $\overline{\omega} \leq A[I-A]^{-1}(Nb) + Nb$, which implies a *non-free lunch* in the initial period. Every national economy has the common consumption space $C \equiv \left\{c \in \mathbb{R}^n_+ \mid c \geqq b\right\} \times [0,1]$ and the common welfare function $u : C \to \mathbb{R}$, defined as follows: for each $(c,l) \in C$,

$$u(c,l) = 1 - l.$$

That is, no nation is concerned about an increase in consumption goods beyond the subsistence level b, but nations evaluate their social welfare in terms of the increase in free hours (leisure time), once b is guaranteed. An *international economy* is thus defined by the profile $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$, which we call a *subsistence* (*international*) *economic environment*.

The time structure of production is such that the capital goods available in the present period of production cannot exceed the amount of capital goods accumulated by the end of the preceding period of production. Furthermore,

(1) Given the market prices $p_{t-1} \geq \mathbf{0}$ at the beginning of period t, each nation $\nu \in \mathcal{N}$ purchases under the constraint of its wealth endowment $p_{t-1}\omega^\nu_t$, capital goods Ax^ν_t as production inputs in the present period. Each nation also purchases commodities δ^ν_t to sell, for speculative purposes, at the end of the present period.

If A is *indecomposable*, then it has at least one non-zero off-diagonal entry in every row and column.

[8] For all vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, 2, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, 2, \ldots, n)$.

(2) Each nation is engaged in production activity of period t by inputting labor Lx_t^ν and the purchased capital goods Ax_t^ν.

(3) The production activity is completed and x_t^ν is the output at the end of this period. Then, in goods markets with market prices $p_t \geq \mathbf{0}$, each nation earns revenue $(p_t x_t^\nu + p_t \delta_t^\nu)$ derived from output x_t^ν, as well as the speculative commodity bundle δ_t^ν. The nation uses this revenue to purchase the bundle b for consumption at the end of this period, and capital stock ω_{t+1}^ν for production in the next period. Therefore, the wealth endowment carried over to the next period $t+1$ is $p_t \omega_{t+1}^\nu$.

Note that the above three-stage time structure of production is assumed to simply describe the aggregated economic actions of a nation $\nu \in \mathcal{N}$, resulting from the optimization decisions of all its citizens. Moreover, though the optimization decisions internal to each nation are not specified, we implicitly assume that each nation $\nu \in \mathcal{N}$ is the representative agent of all its citizens,[9] and citizens exchange their production factors within the territory in domestic factor markets; therefore, labor is traded in a given period t within a given nation at its domestic wage rate w_t^ν, while capital is borrowed and lent within the nation by its domestic interest rate r_t^ν at period t. Thus, in equilibrium with the domestic wage rate and interest rate in period t, $(w_t^\nu, r_t^\nu) \geq (0,0)$, the total net revenue from production activity $p_t x_t^\nu - p_{t-1} A x_t^\nu$ of nation ν is decomposed exhaustively into total wage income $w_t^\nu L x_t^\nu$ and total interest income $r_t^\nu p_{t-1} A x_t^\nu$ through the domestic market.[10]

Therefore, given a price system $\left(\{p_{t-1}, p_t\}; (w_t^\nu, r_t^\nu)_{\nu \in \mathcal{N}}\right)$, in period t, each nation $\nu \in \mathcal{N}$ solves the following (aggregated) optimization program (MP_t^ν):

$$(MP_t^\nu) \quad \min_{x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu \in \mathbb{R}_+^n} l_t^\nu$$

$$\begin{aligned} \text{subject to} \quad p_t x_t^\nu + p_t \delta_t^\nu &\geqq p_t b + p_t \omega_{t+1}^\nu; \\ p_t x_t^\nu - p_{t-1} A x_t^\nu &= w_t^\nu L x_t^\nu + r_t^\nu p_{t-1} A x_t^\nu; \\ l_t^\nu &= L x_t^\nu \leqq 1; \\ p_{t-1} \delta_t^\nu + p_{t-1} A x_t^\nu &= p_{t-1} \omega_t^\nu; \\ p_t \omega_{t+1}^\nu &\geqq p_{t-1} \omega_t^\nu. \end{aligned}$$

[9] The welfare function of subsistence economies is consistent with this representative agent assumption, since it fulfills the homotheticity.

[10] Such a setting of domestic markets is shared with the standard Heckscher-Ohlin model of international trade, as well as the model by Roemer (1982, chapter 1; 1983).

We denote the set of solutions to the optimization program (MP_t^ν) of each nation ν, in period t by $\mathcal{O}_t^\nu\left(\{p_{t-1}, p_t\}; (w_t^\nu, r_t^\nu)_{\nu\in\mathcal{N}}\right)$.

For simplicity, we focus on the subset of equilibria in which prices remain constant over time, i.e., $p_t = p_{t+1} = p^*$. In this case, nations are indifferent to the selection of speculative commodity bundle δ_t^ν, whenever the budget constraint is met. Moreover, in this case, any $\omega_{t+1}^\nu \in \mathbb{R}_+^n$ satisfying $p^*\omega_{t+1}^\nu = p^*\omega_t^\nu$ is an optimal selection, and $p^*x_t^{*\nu} - p^*Ax_t^{*\nu} = p^*b$ holds at $\left(x_t^{*\nu}, \delta_t^\nu, \omega_{t+1}^\nu\right) \in \mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right)$ for any $\delta_t^\nu, \omega_{t+1}^\nu \in \mathbb{R}_+^n$ satisfying $p^*\delta_t^\nu + p^*Ax_t^{*\nu} = p^*\omega_t^\nu = p^*\omega_{t+1}^\nu$. Because of these, we can remove the elements $\delta_t^\nu, \omega_{t+1}^\nu$ from $\mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right)$.

Definition 1: A *reproducible solution* (RS) for a subsistence economy $\langle\mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu\in\mathcal{N}}\rangle$ in period t is a price vector $\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right)$ and an associated profile of actions $(x_t^{*\nu})_{\nu\in\mathcal{N}}$ such that for any t:

(i) for each $\nu \in \mathcal{N}$, $x_t^{*\nu} \in \mathcal{O}_t^\nu\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right)$ holds;

(ii) $Nb + \sum_{\nu\in\mathcal{N}} \omega_{t+1}^\nu \leqq \sum_{\nu\in\mathcal{N}} x_t^{*\nu}$;

(iii) $A\left(\sum_{\nu\in\mathcal{N}} x_t^{*\nu}\right) \leqq \sum_{\nu\in\mathcal{N}} \omega_t^\nu$; and

(iv) $\sum_{\nu\in\mathcal{N}} \omega_{t+1}^\nu \geqq \sum_{\nu\in\mathcal{N}} \omega_t^\nu$.

Definition 1 states that in an RS, taking the price system $\left(p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right)$, every nation chooses its own optimal action in each period (condition (i)); aggregate gross outputs are sufficient to meet aggregate demand of subsistence consumption bundles in each period, and aggregate capital stock invested for the next period (condition (ii)); aggregate activities of production are feasible under the stock of capital goods in each period (condition (iii)); and aggregate capital endowment $\sum_{\nu\in\mathcal{N}} \omega_t^\nu$ in each period is at least reproduced and carried over for production in the next period (condition (iv)). Note that according to conditions (ii), (iii), and (iv) of Definition 1,

(ii') $\sum_{\nu\in\mathcal{N}} x_t^{*\nu} - A\left(\sum_{\nu\in\mathcal{N}} x_t^{*\nu}\right) \geqq Nb$.

Therefore, in every period t, aggregate net outputs are sufficient to meet aggregate demand of the subsistence consumption bundles.

We next show that any allocation at an RS is Pareto efficient. From this property, we can observe that any equilibrium allocation associated with any RS does not entail any capital accumulation; it simply allows the simple reproduction of the initial level of capital stocks $\overline{\omega}$, at each period of production. As a preliminary step, let us define:

Definition 2: Given a subsistence economy without a labor or capital market $\langle \mathcal{N}, (A, L, u), \omega_t \rangle$, $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$, is a *feasible allocation in period* t if and only if:

(1) $\sum_{\nu \in \mathcal{N}} x_t^\nu \geqq Nb + \omega_{t+1}$;

(2) $A \left(\sum_{\nu \in \mathcal{N}} x_t^\nu \right) \leqq \omega_t$;

(3) $\omega_{t+1} \geqq \omega_t$; and

(4) $Lx_t^\nu \in [0, 1]$ $(\forall \nu \in \mathcal{N})$.

That is, a feasible allocation at each period is a profile of each nation's production activity vector $(x_t^\nu)_{\nu \in \mathcal{N}}$, and a vector of commodities ω_{t+1} for replacing as capital used in the next period, satisfying the same conditions as those of Definition 1(ii), (iii), and (iv).

Definition 3: Given a subsistence economy without a labor or capital market $\langle \mathcal{N}, (A, L, u), \omega_t \rangle$, $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$ is a *Pareto efficient allocation in period* t if and only if it is feasible, and there is no other feasible allocation $\left((x'^\nu)_{\nu \in \mathcal{N}}, \omega'_{t+1} \right) \in \mathbb{R}_+^{nN} \times \mathbb{R}_+^n$ such that $L \left(\sum_{\nu \in \mathcal{N}} x_t'^\nu \right) < L \left(\sum_{\nu \in \mathcal{N}} x_t^\nu \right)$.

That is, a feasible allocation at each period is Pareto efficient if and only if there is no other feasible allocation whose aggregate labor supply is less than that of this allocation. Note that the latter condition is equivalent to the standard condition of Pareto non-improvement in subsistence economies. Moreover, given $\overline{\omega} \geqq A [I - A]^{-1} (Nb)$, any Pareto efficient allocation $\left((x_t^\nu)_{\nu \in \mathcal{N}}, \omega_{t+1} \right)$ in period t is characterized by $\omega_{t+1} = \overline{\omega}$, $\left(\sum_{\nu \in \mathcal{N}} x_t^\nu \right) = [I - A]^{-1} (Nb)$, and $L \left(\sum_{\nu \in \mathcal{N}} x_t^\nu \right) = Nvb$, where $v \equiv L [I - A]^{-1}$.

Proposition 1: Given an economy $\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu \in \mathcal{N}} \rangle$ with $\sum_{\nu \in \mathcal{N}} \omega_0^\nu = \overline{\omega}$, let $\langle p; (w_t^\nu, r_t^\nu)_{\nu \in \mathcal{N}}, (x_t^\nu)_{\nu \in \mathcal{N}} \rangle$ be an RS in period t. Then, $(x_t^\nu)_{\nu \in \mathcal{N}}$ is Pareto efficient.

The proofs of all formal results are in the Appendix. Note that, as shown in the proof of Proposition 1, $\omega_{t+1} = \omega_t$ holds for any RS, which implies that capital goods are not accumulated but simply replenished period by period in any RS.

In what follows, we devote special attention to the subset of RSs with *incomplete specialization*, in which each $\nu \in \mathcal{N}$ produces all commodities:[11]

Definition 4: An *RS with incomplete specialization* for a subsistence economy $\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu\in\mathcal{N}} \rangle$ in period t is an RS $\langle p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}, (x_t^{*\nu})_{\nu\in\mathcal{N}} \rangle$, such that for each $\nu \in \mathcal{N}$, $x_t^{*\nu} \in \mathbb{R}_{++}^n$ holds.

At an RS with incomplete specialization, $p^* = (1 + r_t^{\nu*}) p^* A + w_t^{\nu*} L$ holds for every $\nu \in \mathcal{N}$ because $x_t^{*\nu} > \mathbf{0}$. Therefore, $p^* > \mathbf{0}$ since $L > \mathbf{0}$.

The following theorem establishes that factor prices are cross-nationally equalized in equilibria characterized by incomplete specialization.

Theorem 1 [Factor Price Equalization]: Given an economy $\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu\in\mathcal{N}} \rangle$, let $\langle p^*; (w_t^{\nu*}, r_t^{\nu*}), (x_t^{*\nu})_{\nu\in\mathcal{N}} \rangle$ be an RS with incomplete specialization. Suppose that there is at least one pair of $i, j = 1, \ldots, n$, such that $i \neq j$ and $\frac{p^* A \mathbf{e}_i}{L_i} \neq \frac{p^* A \mathbf{e}_j}{L_j}$, where $\mathbf{e}_i$ denotes the i-th unit vector. Then, $(w_t^{\nu*}, r_t^{\nu*}) = \left(w_t^{\nu'*}, r_t^{\nu'*}\right)$ for all $\nu, \nu' \in \mathcal{N}$.

That is, the equalization of wages and interest rates among nations emerges in any RS with incomplete specialization, even though no international market for capital and labor exists. By utilizing this property, the following analysis on the existence of UE will provide a new observation on the UE-generation mechanism.

3 Exploitation as UE

By noting that condition (ii) of Definition 1 is reduced to (ii′), as shown in the previous section, the notion of labor exploitation in subsistence international economies is formally defined as follows:

Definition 5: For any subsistence economy $\left\langle \mathcal{N}, (A, L, u), (\omega_0^\nu)_{\nu\in\mathcal{N}} \right\rangle$, let

[11] For a typical international economic environment with two commodities, the equilibrium notion of incomplete specialization is naturally defined such that each nation produces both commodities. When considering the case with three or more types of commodities, we may have two extensions of the incomplete specialization notion: one that each nation produces all commodities and the other that each nation produces at least two types of commodities. In this paper, we adopt the former extension.

$\left\langle p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}, (x_t^{*\nu})_{\nu\in\mathcal{N}}\right\rangle$ be an RS in period t. Then, the amount of socially necessary labor required to produce b as a net output is

$$\frac{1}{N}L\left(\sum_{\nu\in\mathcal{N}} x_t^{*\nu}\right) = vb = L\left[I - A\right]^{-1} b,$$

where $v = L\left[I - A\right]^{-1}$ is called the *labor value vector*. Moreover, for each nation $\nu \in \mathcal{N}$, the supply of labor hours to earn revenue p^*b for its own survival is $Lx^{*\nu}$, which implies

$$\nu \text{ is an } \textit{exploiting} \text{ nation} \Longleftrightarrow Lx^{*\nu} < vb;$$
$$\nu \text{ is an } \textit{exploited} \text{ nation} \Longleftrightarrow Lx^{*\nu} > vb.$$

Denote the sets of exploiters and exploited respectively by $\mathcal{N}^{ter}$ and $\mathcal{N}^{ted}$.

Definition 6 [Roemer (1982, Definitions 1.3 and 1.4)]**:** For any subsistence economy $\left\langle \mathcal{N}, (A, L, u), (\omega_0^{\nu})_{\nu\in\mathcal{N}}\right\rangle$, an RS in period t, $\left\langle p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}, (x_t^{*\nu})_{\nu\in\mathcal{N}}\right\rangle$, is *inegalitarian* if and only if $\mathcal{N}^{ter} \neq \varnothing$ and $\mathcal{N}^{ted} \neq \varnothing$.

Thus, if an RS in period t is inegalitarian, it involves UE of labor. By contrast, we can state that for any subsistence economy $\left\langle \mathcal{N}, (A, L, u), (\omega_0^{\nu})_{\nu\in\mathcal{N}}\right\rangle$, an RS in period t, $\left\langle p^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}, (x_t^{*\nu})_{\nu\in\mathcal{N}}\right\rangle$ is *egalitarian* if and only if $Lx_t^{*\nu} = Lx_t^{*\nu'}$ for all $\nu, \nu' \in \mathcal{N}$.

Definition 5 presents a standard Okishio (1963)-Morishima (1973) form of exploitation as UE in subsistence economies with simple Leontief production techniques. On the contrary, if a more general class of production economies is considered, many alternative definitions of exploitation have been proposed other than the Okishio-Morishima form, as discussed by Veneziani and Yoshihara (2014; 2015), Yoshihara (2010; 2015), and Yoshihara and Veneziani (2009). However, all such alternative exploitation forms are reduced to Definition 5 within the restricted class of subsistence economies with simple Leontief production techniques. Therefore, the following analysis on the existence and characterization of inegalitarian RSs is free from debate on the proper definitions of labor exploitation.

Though Definition 6 is consistent with the formal definition of Marxian labor exploitation, it may be still unclear on why unequal labor transfers across nations should be considered a manifestation of *exploitation* in the Marxian sense, though Marx (1968, chapter 20, (e)) and Roemer (1982, Chapter 1; 1983) also use this term in the same context of international economic relations. We will return to this question in Section 4.5.

4 Existence and Characterization of Free Trade Equilibria with UE

Here we consider conditions which give rise to *inegalitarian RS*s. To do so, we first classify economies into two types in Section 4.1, according to the existence (or nonexistence) of essential technical differences among sectors. We will also classify them according to their endowment of excessive (or non-excessive) capital stocks. Then, Section 4.2 examines the existence problem of inegalitarian RSs in economies with non-excessive capital stocks, $\overline{\omega} = A\left[I - A\right]^{-1}(Nb)$, and non-essential technical differences among sectors; Section 4.3 examines economies with non-excessive capital stocks and essential technical differences among sectors; and Section 4.4 examines economies with excessive capital stocks, $\overline{\omega} \geq A\left[I - A\right]^{-1}(Nb)$.

In the following discussion, without loss of generality, we remove any subscript "t" whenever RSs are presented. As discussed in the proof of Proposition 1 (see the Appendix below), the equilibrium price vector p should be positive with $p - pA > \mathbf{0}$ at an RS, and its associated social production activity vector is $x^* \equiv (I - A)^{-1}(Nb) > \mathbf{0}$. Let $\frac{1}{1+R}$ with $R > 0$ be the Frobenius eigenvalue of the productive and indecomposable matrix A. Here $R > 0$ follows from the productiveness of A.

4.1 Classification of Economies with respect to Technical Difference among Sectors

Let us classify economies based on whether essential technical differences among sectors exists or not. It is given by examining whether the Frobenius eigenvector of the Leontief matrix A and the vector of the labor input coefficients L are linearly independent or not.

Lemma 1: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$, let a price vector $p^r = (1 + r)p^r A + wL > \mathbf{0}$ be associated with its unique equal interest rate $r > 0$. Then, p^r is the unique Frobenius eigenvector of A associated with the

Frobenius eigenvalue $\frac{1}{1+R}$ such that p^r is proportional to the vector of labor values v if and only if $p^r A$ and L are linearly dependent. By contrast, let $\overline{p} = (1+R)\,\overline{p}A > \mathbf{0}$ be the unique Frobenius eigenvector of A associated with the Frobenius eigenvalue $\frac{1}{1+R}$. Then, for any $r' \in [0, R]$, $\overline{p}$ is the price vector associated with the equal interest rate r', that is, $\overline{p} = (1+r')\,\overline{p}A + \overline{w}L$ holds for some $\overline{w} > 0$ and is proportional to the vector of labor values v if and only if $\overline{p}A$ and L are linearly dependent.

Lemma 1 suggests that if in an economy with the Leontief production technique (A, L), the unique Frobenius eigenvector $\overline{p}$ of A is linearly independent of the vector L, then for any equilibrium price vector $p^r \equiv L\,(I - (1+r)\,A)^{-1}$ associated with an equal interest rate $r \in (0, R)$, $p^r A$ and L are linearly independent.

Lemma 1 has some interesting implications. First, it classifies subsistence economies with Leontief production techniques into two types. One type are economies in which the unique Frobenius eigenvector $\overline{p}$ of A and L are linearly dependent. In such a case, if every nation establishes a positive wage rate in its domestic factor market under the international equilibrium, a common capital-labor ratio is established among all sectors, evaluated by the corresponding equilibrium prices of commodities. Thus, in this type of economy, *essentially no technical differences among sectors* exists in that the capital-labor ratios are common among sectors under all equilibria. The other type are economies in which the unique Frobenius eigenvector $\overline{p}$ of A and L are linearly independent. Here, *technical differences among sectors* exist, in that the capital-labor ratios are not identical among sectors under some equilibrium.

Second, in combination with Theorem 1, Lemma 1 offers the following observation. If the production technique (A, L) reveals that its unique Frobenius eigenvector $\overline{p}$ and L are linearly dependent, then no price vector is associated with an equal positive interest rate, except in cases of labor value pricing (i.e., when the price vector is proportional to the vector of labor value).

4.2 Egalitarian RSs in Economies with no Technical Difference among Sectors

Given non-excess capital stocks $\overline{\omega} = A\,[I - A]^{-1}\,(Nb)$, let us examine the existence and characterization of RSs in economies with no essential technical differences among sectors. Let us take any subsistence economy with a Leontief production technique such that its unique Frobenius eigenvector and its labor coefficient vector are linearly dependent. In such cases, at most only egalitarian RSs exist.

Theorem 2: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A[I - A]^{-1}(Nb)$, at most, only an *egalitarian RS* exists under the equal initial endowments of financial capital, and no inegalitarian RS exists under any initial endowments of financial capital if and only if the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$ of A and $L > \mathbf{0}$ are linearly dependent.

Theorem 2, combined with Lemma 1, suggests that in any subsistence economy with no essential technical differences among sectors, the only available type of free trade equilibrium is that of egalitarian RSs, which is realized under the equal initial distribution of financial capital, and the equilibrium prices of all such RSs are characterized by labor value pricing.

Note that Theorem 2 also suggests that international trade of commodities could be conducted among nations under egalitarian RSs, although no nation can enjoy a strict gain from trade under such RSs. In such RSs, every nation can choose an autarkic economy in that, if preferred, it would self-produce and consume the net output b, by investing equally distributed financial capital and equalized labor supply vb, which is, for every nation, indifferent to its own production activities currently implemented under such RSs. Therefore, no nation has a strong rationale to shift from autarkic activity to free trade equilibrium.

4.3 Inegalitarian RSs in Economies with Technical Difference among Sectors

In the following argument, our main concern focuses on economies with technical differences among sectors, in which the existence and characterization of RSs are examined. Let us take any subsistence economy with a Leontief production technique (A, L) such that its unique Frobenius eigenvector and its labor coefficient vector are linearly independent. From Lemma 1, this case is equivalent to the case that for any $r \in (0, R)$, its associated price vector $p^r = L(I - (1 + r)A)^{-1}$ has the property that p^r and L are linearly independent, which is equivalent to the property that $p^r(I - A)$ and $p^r A$ are linearly independent. Therefore, Theorem 1 suggests that in incompletely specialized RSs, every nation's factor prices must be equalized, meaning that to examine the existence of equilibrium price vectors in such equilibria, it is sufficient to focus only on the types of $p^r = L\left(I-(1+r)A\right)^{-1}$.

Let $\overline{\omega} = A\left[I-A\right]^{-1}(Nb)$. For each $r \in (0,R)$, let $p^r \equiv L\left(I-(1+r)A\right)^{-1}$. We would like to identify the domain of wealth distributions over which RSs with incomplete specialization can exist. To do so, first, we will identify the minimal value, say θ^r, of individual wealth shares with respect to the price vector p^r, such that an agent with this share of wealth can find a feasible action. Here, the feasible action implies that this agent can purchase the subsistence bundle at price p^r by supplying some labor hours equal to or less than unity. Second, by varying $r \in (0,R)$, we will identify the minimal value, say $\underline{\theta}$, of the wealth sharing among all values in $\{\theta^r \mid r \in (0,R)\}$. Then, we will show that any distribution of wealth $(W^\nu)_{\nu\in N} \in \mathbb{R}^N_+$ ensures the existence of incompletely specialized RSs if the minimal share rate within this distribution is larger than $\underline{\theta}$.

Let us discuss these steps formally. For each given $\theta \in [0,1]$, consider a non-negative and non-zero vector $x \in \mathbb{R}^n_+$ in order to solve the following system of equations:

$$\left\{\begin{array}{c} p^r A x = \theta p^r \overline{\omega}; \\ p^r\left(I-A\right)x = p^r b; \\ Lx \in [0,1]. \end{array}\right.$$

Denote the set of solutions for this system of equations by $X^r(\theta)$ with the generic element $x^r(\theta)$. Then, define the following program:

$$\min_{\theta\in[0,1]} \theta, \text{ subject to } X^r(\theta) \neq \varnothing. \ (*)$$

Note that $\cup_{\theta\in[0,1]} X^r(\theta)$ is non-empty, since for $\theta = \frac{1}{N}$, $x(\theta) = (I-A)^{-1} b$ is the solution. Denote the solution of the program (*) by θ^r.

Then, we can define $\underline{\theta} \equiv \inf_{r\in(0,R)} \theta^r$. Let $\triangle(\overline{\omega}) \equiv \left\{p \in \mathbb{R}^n_+ \mid p\overline{\omega} = 1\right\}$ and

$$\triangle_{\underline{\theta}}(W) \equiv \left\{(W^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^N_{++} \mid \min_{\nu\in\mathcal{N}} W^\nu > \underline{\theta}, \ \sum_{\nu\in\mathcal{N}} W^\nu = 1\right\}.$$

Finally, for notational convenience, let

$$\overline{\Omega} \equiv \left\{(\omega^\nu)_{\nu\in N} \in \mathbb{R}^{nN}_+ \mid \sum_{\nu\in\mathcal{N}} \omega^\nu = \overline{\omega} \ \& \ LA^{-1}\omega^\nu \in [0,1] \quad (\forall \nu \in \mathcal{N})\right\}.$$

Now, we are ready to show the existence and characterization of inegalitarian RSs.

Theorem 3: Let an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A [I - A]^{-1} (Nb)$ be such that the unique Frobenius eigenvector of A and $L > \mathbf{0}$ are linearly independent. Then, for any profile $(W^{\nu})_{\nu \in \mathcal{N}} \in \triangle_{\underline{\theta}} (W)$ of the initial endowments of financial capital, there exist $(p^{*}, w^{*}, r^{*}) \in \triangle (\overline{\omega}) \times \mathbb{R}_{++} \times \mathbb{R}_{+}$ and $(\omega^{\nu})_{\nu \in \mathcal{N}} \in \overline{\Omega}$ by which $\left\langle p^{*}; (w^{*}, r^{*}), (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ constitutes an RS with incomplete specialization for the subsistence private ownership economy $\left\langle \mathcal{N}, (A, L, u), (\omega^{\nu})_{\nu \in \mathcal{N}} \right\rangle$, such that the following property is satisfied: this RS is inegalitarian if and only if the profile $(W^{\nu})_{\nu \in \mathcal{N}}$ features unequal initial endowments of financial capital across nations and the equilibrium price vector $p^{*} > \mathbf{0}$ is associated with a positive equal interest rate $r^{*} > 0$.

Theorem 3 implies that in economies with an essential technical differences among sectors, an RS exists in a broad class of initial distributions of financial capital. This result is in sharp contrast to the case of economies with no technical differences among sectors, where only the equal distribution of financial capital allows the existence of RSs. In addition, owing to the broadly available domain of initial distributions, most RSs are characterized as free trade equilibria derived from the unequal initial distribution of financial capital. More interestingly, such RSs are characterized as having international division of labor because of the unequal distribution of financial capital. That is, relatively rich nations are more specialized toward capital-intensive production activities, while relatively poor nations are more specialized toward labor-intensive production activities, in that for any $\nu, \nu' \in \mathcal{N}$ with $W^{\nu} > W^{\nu'}$, $\frac{p^{*} A x^{*\nu}}{L x^{*\nu}} > \frac{p^{*} A x^{*\nu'}}{L x^{*\nu'}}$ holds under the RS $\left\langle p^{*}; (w^{*}, r^{*}), (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$.

Such RSs are also characterized as involving UE, whenever their associated interest rates are positive. In particular, relatively rich nations supply fewer labor hours than the socially necessary labor hours vb to produce the subsistence vector b, meaning that they are exploiting, while relatively poor nations supply more labor hours than the socially necessary labor hours vb, meaning that they are being exploited. In addition, such RSs do not involve UE whenever interest rates are zero.

Finally, in RSs with a zero interest rate, the international division of labor occurs due to the unequal distribution of financial capital, even though labor supply is equalized among nations. Note that a zero interest rate yields no net reward for investing capital goods into the production process. Then, more capital-intensive activities cannot reduce labor supply for the nation, and accordingly, labor supply is equalized among nations. Moreover, RSs with a zero interest rate are also characterized by labor value pricing.

4.4 Egalitarian RSs in Economies with Excessive Capital Endowments

Since all arguments presented above assume $\overline{\omega} = A[I-A]^{-1}(Nb)$, let us examine the case of $\overline{\omega} \geq A[I-A]^{-1}(Nb)$. Note, if $\overline{\omega} \ngeqq A[I-A]^{-1}(Nb)$, there is no RS in subsistence economies. Therefore, the only remaining task in characterizing the class of RSs is to check the case of $\overline{\omega} \geq A[I-A]^{-1}(Nb)$.

Theorem 4: Let an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ be such that $\overline{\omega} \geq A[I-A]^{-1}(Nb)$. Then, for any RS $\left\langle p^*; (w^{\nu *}, r^{\nu *})_{\nu \in \mathcal{N}}, (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ associated with a suitable $(\omega^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$, $p^* = \varsigma v$ holds for some $\varsigma > 0$, and there is no exploitation.

Theorem 4 implies that in any subsistence economy with $\overline{\omega} \geq A[I-A]^{-1}(Nb)$, no RS has exploitation and its equilibrium commodity prices are characterized by labor value pricing. This property holds regardless of whether the equilibrium interest rates are positive. It is particularly interesting when $\overline{p}$ and L are linearly dependent, since in such a case, an egalitarian RS with a positive equilibrium interest rate can exist even under a suitable unequal distribution of wealth, as the RS $\left\langle \overline{p}; (w, r), \left(\frac{x^*}{N}, \ldots, \frac{x^*}{N}\right) \right\rangle$ with $r > 0$ discussed in the proof of Theorem 4 (see the Appendix below).

4.5 Indeterminacy of Inegalitarian RSs by Competitive Markets

In summary, Theorems 2–4 together imply that regardless of whether economies involve inter-sector technical heterogeneity, any free trade equilibrium involves exploitation if and only if this equilibrium is derived from an unequal distribution of wealth, and the corresponding equilibrium prices deviate from the labor value pricing.

Corollary 1: For any economy $\left\langle \mathcal{N}, (A, L, u), (\omega^\nu)_{\nu \in \mathcal{N}} \right\rangle$ with $\overline{\omega} \geqq A[I-A]^{-1}(Nb)$, and any RS $\left\langle (p^*, w^*, r^*), (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ of this economy, this RS is inegalitarian, if and only if, $(p^* \omega^\nu)_{\nu \in \mathcal{N}}$ is unequal and there is no $\varsigma > 0$, such that $p^* = \varsigma v$.

In other words, even if an equilibrium price vector is associated with a positive interest rate, it does not involve exploitation under the unequal distribution of wealth whenever equilibrium prices are proportional to the labor values. Such a situation is possible, according to Theorems 3 and 4. Therefore, Corollary 1 implies that positive equilibrium interest rates and the existence of exploitation are not necessarily equivalent—even under the unequal distribution of wealth.

Note that if a subsistence international economy entails non-excessive capital stocks and essential technical differences among sectors, Theorem 3 and Corollary 1 suggest that the equilibrium transfer of labor across nations is generically indeterminate, whereas, if the economy does not entail both, Theorems 2 and 4 suggest that the exchange of labor among nations is equalized in equilibrium. This indeterminacy implies that even if there are rich and poor nations with a strong motivation to interact via competitive commodity markets to gain from the international division of labor, these institutions alone are insufficient for the realization of an inegalitarian RS: the egalitarian RS may instead emerge through the competitive trade of commodities between rich and poor.

In this situation, let us consider a *counterfactual bargaining among nations* as one of the possible determinants for the selection of an inegalitarian RS and for eliminating the egalitarian RS. For simplicity, let us focus on the case of bargaining between two nations, the rich ν and the poor μ. Assume that $\omega^\nu \geq A[I-A]^{-1} b \geq \omega^\mu$ and $\omega^\nu + \omega^\mu = A[I-A]^{-1}(2b)$. Therefore, ν is sufficiently rich that it can survive under an autarkic economy, while μ is sufficiently poor that it cannot survive under the autarchy economy. Consider a Nash bargaining problem between these two, where if both parties cannot reach an agreement on the selection of a proper RS, they must exit international trade and return to autarkic economic conditions. In the autarkic allocation, ν can have $(c^\nu, l^\nu) = (b, vb)$, that is, consuming b and supplying vb-labor hours, whereas μ simply dies because it cannot produce the bundle b through its own autarkic production. In the option set of possible RSs, the egalitarian RS associated with the allocation of labor and consumption, $((c^{*\nu}, l^{*\nu}), (c^{*\mu}, l^{*\mu})) = ((b, vb), (b, vb))$, is most preferable for the poor μ, while there is an inegalitarian RS associated with the allocation, $((c^{**\nu}, l^{**\nu}), (c^{**\mu}, l^{**\mu})) = ((b, 2vb-1), (b, 1))$, where $vb < 1 < 2vb$, in which

ν enjoys the highest level of welfare among the options of RSs.[12][13] Thus, this bargaining problem is given by a pair (S, d) of the utility possibility set S and the disagreement point d, where S is defined as the convex-hull of the four utility allocations $\overline{u}^{*} = (u(b, vb), u(b, vb))$, $\overline{u}^{**} = (u(b, 2vb-1), u(b, 1))$, $\overline{u}^{***} = (u(b, 2vb-1), u(0, 0))$, and $d = (u(b, vb), u(0, 0))$.[14] Note that $u(0, 0)$ represents the utility level of dying.

In this bargaining problem, ν can enjoy a stronger bargaining power than μ due to the inequality of capital endowments between them, and since ν can take advantage of this asymmetric power structure, the egalitarian RS can never emerge as an equilibrium bargaining outcome. Indeed, none of the representative bargaining solutions like the Nash, the Kalai-Smorodinsky, and the egalitarian solutions can select the egalitarian RS as the predictable outcome. For the rich ν is indifferent between the egalitarian RS-allocation $((b, vb), (b, vb))$ and the disagreement point allocation $((b, vb), (0, 0))$, and even if the egalitarian RS is offered as their agreement, ν can reject this proposal. In contrast, if an inegalitarian RS is offered as agreement, μ cannot but accept this offer since the autarkic allocation resulting from the breakdown of bargaining would force μ to die. In other words, any bargaining solution selecting the egalitarian RS must violate the *strong individual rationality* condition.

Though such bargaining is counterfactual, the logical consequence of such a hypothetical scenario may give us a reason to infer that an egalitarian RS is unlikely to emerge even though it logically exists. Furthermore, it also suggests that the existence of an asymmetric power structure between rich and poor nations would be indispensable for the logical generation of UE-exploitation in free trade equilibria.[15]

[12] Remember that 1 is the maximum length of labor hours available to all nations.

[13] Note that in this inegalitarian RS, the equilibrium wage rate is insufficient for purchasing the subsistence bundle, $w < pb$. This implies that in both nations, workers without property cannot survive by their wage revenue alone, so that income redistribution policies must be implemented.

[14] We adopt the Nash bargaining problem as a technically convenient reduced-form approach to some non-cooperative bargaining game. As is common in the literature on non-cooperative implementation of bargaining solutions, all the representative bargaining solutions should be understood as the equilibrium outcome of some underlying noncooperative bargaining procedure.

[15] As Roemer (1982, chapter 1, pp. 44–45) points out, a (re)switching of the position in wealth distribution between ν and μ may occur, according to which equilibrium price vector is realized, unless $\omega^{\nu} \geq A[I-A]^{-1} b \geq \omega^{\mu}$ holds. In this case, we cannot observe a

In addition, this scenario of counterfactual bargaining may give us a reasonable explanation of why UE-transfer of labor across nations should be considered exploitative. As argued in the recent literature on exploitation theory by Cohen (1995), Wright (2000), Vrousalis (2013), Veneziani (2013), and Yoshihara (2015), the UE-transfer of labor among agents per se is necessary, but the existence of an asymmetric power structure should also be observed to diagnose such a UE-transfer as exploitative. In this respect, the above bargaining scenario suggests the existence of an asymmetric structure of bargaining power between rich and poor nations, which leads them to equilibrium with the UE-transfer of labor from the poor to the rich.

Indeed, inegalitarian RSs observed in Theorem 3 can be worthwhile to manifest as exploitative, according to Wright's (2000) criteria. Wright (2000) defines that exploitation exists if (a) *the inverse interdependent welfare principle*,[16] (b) *the exclusion principle*,[17] and (c) *the appropriation principle*[18] are satisfied. We can test whether the relationship between the rich and poor nations in inegalitarian RSs satisfies the three criteria. First, criterion (a) is satisfied, since in subsistence international economies, the richer nation under an inegalitarian RS enjoys better welfare by working shorter hours relative to the egalitarian RS, while the poorer nation lowers welfare by working longer hours relative to the egalitarian RS. This situation crucially depends on the existence of the poorer nation, which is excluded from access to capital stocks sufficient for autarkic survival. It is only the richer nation, which can specialize in more capital-intensive activities via the international division of labor due to the existence of the poorer nation, which can only specialize in more labor-intensive activities. Indeed, if the poorer nation exited the world economy, the richer nation would have to expend the socially necessary labor time vb to access the subsistence bundle, even though the richer nation may

coherent asymmetric power structure of bargaining between ν and μ, since the realization of equilibrium price vectors is indeterminate, and no information is available on who is richer or poorer in advance of the bargaining game. In such a situation, where ν becomes richer at one inegalitarian RS while μ at another inegalitarian RS, it seems ambiguous if the relationship between ν and μ is worth diagnosing as exploitative. This is why we focus on of $\omega^{\nu} \geq A\left[I-A\right]^{-1}b \geq \omega^{\mu}$ here.

[16] That is, material welfare of exploiters causally depends upon the reduction of material welfare of the exploited.

[17] That is, this inverse interdependence of the welfare of exploiters and the exploited depends upon the exclusion of the exploited from access to certain productive resources.

[18] That is, the exclusion generates a material advantage for exploiters because it enables them to appropriate the labor effort of the exploited.

be able to monopolize the most of socially endowed capital stocks. Thus, criterion (b) is satisfied. Finally, it follows from the above arguments that criterion (c) is satisfied in every inegalitarian RS.[19]

5 Concluding Remarks

This paper introduced subsistence international economies with Leontief production techniques and examined the necessary and sufficient condition for the existence of equilibria with UE, namely inegalitarian RSs. First, the findings showed that if the social endowments of aggregate capital goods are excessive relative to the minimal level necessary for the survival of the economy, then no inegalitarian RS exists, regardless of whether the distribution of wealth among nations is unequal. Second, if the social endowments of aggregate capital goods are equal to the minimal level necessary for the survival of the economy, then inegalitarian RSs exist only under the condition that wealth distribution is unequal, and essential technical differences exist among sectors. Such a condition implies that each nation has a strong motivation to participate in international trade based on the principle of comparative advantage. In other words, a richer nation finds its comparative advantage when selecting a more capital-intensive production activity, while a poorer nation finds its comparative advantage when selecting a more labor-intensive production activity. In summary, the existence of inegalitarian RSs is characterized by the unequal distribution of wealth among nations and deviation from the labor value pricing of commodities. These main results suggest that the generation of inegalitarian RSs cannot be ensured by the competitive mechanism of markets under unequal distribution of wealth alone, and an additional scheme of bargaining between richer and poorer nations may solve this indeterminacy.

This characterization demonstrates an interesting contrast with the *Fundamental Marxian Theorem* (FMT) (Okishio, 1963; Morishima, 1973), which shows that the unequal distribution of financial capital and the positivity

[19]Note also that inegalitarian RSs would also be worthwhile to be exploitative, according to Vrousalis' (2013) definition of *economic exploitation*. Indeed, poorer nations without their autarkic survivability are probably *economically vulnerable* to richer nations. Thus, richer nations would regard bargaining with these poorer nations as the process through which the rich *instrumentalizes* the poor's economic vulnerability to *appropriate (the fruit of) the poor's labor*. For further discussion on Vrousalis's (2013) notion of economic exploitation, see Yoshihara (2015).

of profits are necessary and sufficient for the existence of exploitation in economies with labor markets. Unlike the FMT, this paper shows that UE is not generated in equilibrium with a common capital-labor ratio among sectors, even if the equilibrium interest rate is positive and wealth distribution among nations is unequal. However, this does not necessarily imply the violation of FMT in subsistence international economies, for two reasons. First, the premise of the FMT is based on economies with labor markets, while the characterization of inegalitarian RSs in this paper was established in international economies without international labor markets. Second, the FMT discusses the aggregate exploitation rate of the whole working class (the class of agents endowed with no financial capital), while the characterization of inegalitarian RSs in this paper could not refer to the exploitation status of nations with no financial capital, since such nations cannot survive in subsistence international economies. Thus, the main theorems of this paper do not satisfy the premise of the FMT.

The analysis of the existence of UE in the free trade equilibria presented herein is concerned only with the *temporary* features of international trade. However, if *intertemporal* features of international trade were also considered,[20] the existence of UE in dynamic free trade equilibria would have a quite different characterization. Indeed, our companion paper, Yoshihara and Kaneko (2014), shows that in subsistence international economies with infinite horizons, whenever an essential technical differences exists among sectors, inegalitarian RSs generically exist in every period, regardless of whether the initial endowments of aggregate capital goods are excessive.[21] By contrast, in subsistence economies with finite horizons and no discount factor—as in Veneziani (2007; 2013)—there is no inegalitarian RS, regardless of whether the initial distribution of financial capital is unequal, whenever the initial endowments of aggregate capital goods are excessive.

Note also that this paper focused on international trade with incomplete specialization in Leontief production economies with no option of technical

[20]Veneziani (2007; 2013) addresses the issue of the persistent existence of UE-exploitation in intertemporal subsistence economies with labor markets.

[21]More specifically, assuming the maximization of the discounted sum of one-period welfare as each nation's intertemporal optimal decision, we find that under equilibria, the aggregate capital accumulation path $\{\omega_t\}_{t=0}^{\infty}$, starting with ω_0, must fulfill $\omega_t = A\left[I - A\right]^{-1}(Nb)$ for all $t = 1, 2, \ldots$, even if $\omega_0 \geq A\left[I - A\right]^{-1}(Nb)$. That is, the social endowments of aggregate capital goods necessarily converges to the minimal level necessary for survival at each period.

choice. In such an environment, the factor price equalization holds, as shown in Theorem 1. However, it is well known that once a model is extended to allow technical choice, the factor price equalization may not hold in general, as Metcalfe and Steedman (1972; 1973) and Kurose and Yoshihara (2015) discuss. Therefore, if subsistence international economies with an option set of multiple Leontief production techniques are considered, we could not develop our analysis by relying on the factor price equalization as this paper does, meaning that a new analytical technique for the subject would be necessary. This interesting question remains for future research.

6 Appendix: Proofs

Proof of ***Proposition 1.*** As shown by Roemer (1982, chapter 1), $0 < Lx_t^\nu$ holds for any $\nu \in \mathcal{N}$, since there is no labor market. From the definition of (MP_t^ν), $p\omega_{t+1}^\nu = p\omega_t^\nu$ and $px_t^\nu - pAx_t^\nu = pb$ hold, as argued when (MP_t^ν) is defined. Therefore, $p\left[I-A\right]\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) = Npb$ and $p\omega_{t+1} = p\omega_t$ hold. According to Definition 1, $\sum_{\nu\in\mathcal{N}} x_t^\nu - A\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) \geqq Nb$, meaning that $\sum_{\nu\in\mathcal{N}} x_t^\nu \geqq \left[I-A\right]^{-1}(Nb) > \mathbf{0}$ because $\left[I-A\right]^{-1} > \mathbf{0}$. Then, from $px_t^\nu - pAx_t^\nu = pb$ for any $\nu \in \mathcal{N}$, $p\left[I-A\right] > \mathbf{0}$. (Indeed, if $p_j - pA\mathbf{e}_j \leqq 0$, where $\mathbf{e}_j$ denotes the j-th unit vector, for some commodity j, then $x_{jt}^\nu = 0$ holds for any $\nu \in \mathcal{N}$ by optimality, which is a contradiction.) Thus, $p > \mathbf{0}$ because $\left[I-A\right]^{-1} > \mathbf{0}$. Then, $\omega_{t+1} = \omega_t = \overline{\omega}$ holds according to Definition 1(iv), and from $p\left[I-A\right]\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) = Npb$, $\left[I-A\right]\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) = Nb$ holds, meaning that $\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) = \left[I-A\right]^{-1}(Nb)$ and $L\left(\sum_{\nu\in\mathcal{N}} x_t^\nu\right) = Nvb$. This finding implies that $\left(x_t^\nu\right)_{\nu\in\mathcal{N}}$ is Pareto efficient. □

Proof of ***Theorem 1.*** Note that at an RS with incomplete specialization, we have for each $\nu \in \mathcal{N}$,

$$p^*\left[I-A\right] = r_t^{\nu*} p^* A + w_t^{\nu*} L.$$

From the proof of Proposition 1, we know that $p^*\left[I-A\right] > \mathbf{0}$ and $p^* > \mathbf{0}$, which also implies that $p^*A > \mathbf{0}$ by indecomposability of A. Then, we obtain that, for each $\nu, \nu' \in \mathcal{N}$,

$$\left(r_t^{\nu*} - r_t^{\nu'*}\right) p^* A + \left(w_t^{\nu*} - w_t^{\nu'*}\right) L = \mathbf{0}.$$

Take $i, j = 1, \ldots, n$ such that $i \neq j$ and $\frac{p^* A\mathbf{e}_i}{L_i} \neq \frac{p^* A\mathbf{e}_j}{L_j}$. The above system of equations implies that for each $\nu, \nu' \in \mathcal{N}$,

$$\left(r_t^{\nu *} - r_t^{\nu' *}, w_t^{\nu *} - w_t^{\nu' *}\right) \begin{bmatrix} p^* A\mathbf{e}_i & p^* A\mathbf{e}_j \\ L_i & L_j \end{bmatrix} = (0, 0),$$

where L_i denotes the i-th element of $L > \mathbf{0}$. Since $\frac{p^* A\mathbf{e}_i}{L_i} \neq \frac{p^* A\mathbf{e}_j}{L_j}$ for these i, j, we have $p^* A\mathbf{e}_i \cdot L_j - p^* A\mathbf{e}_j \cdot L_i \neq 0$. Then, the matrix $\begin{bmatrix} p^* A\mathbf{e}_i & p^* A\mathbf{e}_j \\ L_i & L_j \end{bmatrix}$ is non-singular, and hence the row vectors $(p^* A\mathbf{e}_i, p^* A\mathbf{e}_j)$ and (L_i, L_j) are linearly independent. Thus, $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu' *}, w_t^{\nu' *}\right)$. Note that this result follows with respect to each $\nu, \nu' \in \mathcal{N}$. Therefore, by fixing i, j, we have that for each $\nu'' \in \mathcal{N} \setminus \{\nu'\}$,

$$\left(r_t^{\nu *} - r_t^{\nu'' *}, w_t^{\nu *} - w_t^{\nu'' *}\right) \begin{bmatrix} p^* A\mathbf{e}_i & p^* A\mathbf{e}_j \\ L_i & L_j \end{bmatrix} = \mathbf{0},$$

which implies $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu'' *}, w_t^{\nu'' *}\right)$. Thus, $(r_t^{\nu *}, w_t^{\nu *}) = \left(r_t^{\nu' *}, w_t^{\nu' *}\right)$ for all $\nu, \nu' \in \mathcal{N}$. □

*Proof of **Lemma 1**.* Let $p^r = (1 + r) p^r A + wL > \mathbf{0}$ be such that $p^r A$ and L are linearly dependent. This finding implies that there exists $\varsigma > 0$ such that $p^r A = \varsigma L$. Therefore, $p^r (I - A) = (r\varsigma + w) L$. Thus, $p^r = (r\varsigma + w) v$, which implies that p^r is proportional to the vector of labor values v. In addition, it follows that $p^r (I - A) = (r + w\varsigma^{-1}) p^r A$, meaning that $p^r (I - (1 + r + w\varsigma^{-1}) A) = \mathbf{0}$. Therefore, since A is indecomposable, $p^r > \mathbf{0}$ is the Frobenius eigenvector of A unique up to scale, and $[1 + (r + w\varsigma^{-1})]^{-1}$ can be the Frobenius eigenvalue of A. By contrast, if $p^r A$ and L are linearly independent, then the vectors p^r and $p^r A$ must be linearly independent. Then, it is impossible to have $p^r = (1 + R) p^r A$ for some $(1 + R) > 0$, which implies that p^r can never be the Frobenius eigenvector of A.

Let $\overline{p} = (1 + R) \overline{p} A > \mathbf{0}$ be the unique Frobenius eigenvector of A associated with the Frobenius eigenvalue $\frac{1}{1+R}$ such that $\overline{p} A$ and L are linearly dependent. Therefore, there exists $\varsigma > 0$ such that $\overline{p} A = \varsigma L$. Then, for any $r' \in [0, R)$, $(R - r') \overline{p} A = (R - r') \varsigma L$ holds, meaning that $\overline{p} = (1 + r') \overline{p} A + \overline{w} L$ for $\overline{w} \equiv (R - r') \varsigma > 0$. Since A is indecomposable, $L (I - (1 + r') A)^{-1} > \mathbf{0}$ exists such that $\overline{p} = \overline{w} L (I - (1 + r') A)^{-1}$ holds.

Moreover, since $\overline{p}(I-A) = R\overline{p}A = R\varsigma L$, $\overline{p} = R\varsigma v$ holds, so that $\overline{p}$ is proportional to the vector of labor values v. By contrast, if $\overline{p}A$ and L are linearly independent, it is impossible to have $\overline{p} = (1+r')\,\overline{p}A + \overline{w}L$ for some $r' \in [0, R)$ and some $\overline{w} > 0$, since $\overline{p}$ and $\overline{p}A$ are linearly dependent by definition. $\square$

Let $\triangle(\overline{\omega}) \equiv \left\{p \in \mathbb{R}^n_+ \mid p\overline{\omega} = 1\right\}$ for $\overline{\omega} \equiv A[I-A]^{-1}(Nb)$ and let $\triangle(W) \equiv \left\{(W)_{\nu\in\mathcal{N}} \in \mathbb{R}^N_{++} \mid \sum_{\nu\in\mathcal{N}} W^\nu = 1\right\}$.

*Proof of **Theorem 2.*** Given the technology (A, L), let us consider $\overline{p}(I-(1+R)A) = \mathbf{0}$, where $\frac{1}{1+R}$ is the unique Frobenius eigenvalue of A and $\overline{p} > \mathbf{0}$ is its associated eigenvector uniquely up to scale. Therefore, let us suppose that $\overline{p} \in \triangle(\overline{\omega})$. By definition, the row vectors $\overline{p}$ and $\overline{p}A$ are linearly dependent.

Then, if the two row vectors $\overline{p}A$ and L are linearly dependent, which is derived from the linear dependency of $\overline{p}$ and L, then a pair of $\overline{p}$ and any allocation $(x^\nu)_{\nu\in\mathcal{N}}$ satisfying $(I-A)\left(\sum_{\nu\in\mathcal{N}} x^\nu\right) = Nb$ and $Lx^\nu = vb$ can constitute an egalitarian RS in an economy with equal initial endowments of financial capital $(W^\nu)_{\nu\in\mathcal{N}} = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$, and no other RS is in an economy with any initial endowments of financial capital. This situation occurs because $\overline{p}(I-A)$ and $\overline{p}A$ are linearly dependent and thus the hyperplanes $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ and $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}Ax = \frac{\overline{p}\overline{\omega}}{N}\right\}$ coincide. Therefore, if wealth endowments are unequal, there is at least one nation $\nu \in \mathcal{N}$ such that $W^\nu < \frac{1}{N}$, meaning that this agent's set $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}Ax \leqq W^\nu\right\}$ of capital-constrained feasible activities is included in the strictly lower contour set $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x < \overline{p}b\right\}$, which implies that this nation has no feasible production activity. By contrast, since $\overline{p}(I-A)$ and L are linearly dependent, the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ and an indifference surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = L(I-A)^{-1}b\right\}$ coincide. Therefore, any point in $\left\{x \in \mathbb{R}^n_+ \mid \overline{p}(I-A)x = \overline{p}b\right\}$ can constitute a solution to the optimization program (MP^ν_t) for any nation under the equal initial endowments of financial capital $(W^\nu)_{\nu\in\mathcal{N}} = \left(\frac{1}{N}, \dots, \frac{1}{N}\right) \in \triangle(W)$, which implies that any nation can realize $Lx^\nu = vb$ as its optimal labor supply. Therefore, $\left(\overline{p}, (x^\nu)_{\nu\in\mathcal{N}}\right) \in \triangle(\overline{\omega}) \times \mathbb{R}^{nN}_+$ with $(I-A)\left(\sum_{\nu\in\mathcal{N}} x^\nu\right) = Nb$ and $Lx^\nu = vb$ for all $\nu \in \mathcal{N}$ can constitute an egalitarian RS in an economy with $(W^\nu)_{\nu\in\mathcal{N}} = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$, and there is no inegalitarian RS in such an economy.

Moreover, if $\overline{p}$ and L are linearly dependent, Lemma 1 implies that $\overline{p}$ can be any equilibrium price vector associated with any equal interest rate $r \in [0, R)$, which is proportional to the labor value vector v. In combination with the previous analysis, this finding further implies that in an economy with the linear dependency of $\overline{p}$ and L, the only available types of RSs are egalitarian associated with the equal initial endowments of financial capital, regardless of whether the associated equal interest rate is positive.

Next, let $\overline{p}$ and L be linearly independent. Then, $\overline{p}A$ and L are linearly independent, since $\overline{p}$ and $\overline{p}A$ are linearly dependent by definition. In this case, no RS corresponds to the price system $\overline{p}$ because no nation's optimal solution can constitute a feasible allocation. First, if wealth endowments are unequal, there is at least one nation $\nu \in \mathcal{N}$ such that $W^{\nu} < \frac{1}{N}$, meaning that this agent's set of capital-constrained feasible activities, $\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}Ax \leqq W^{\nu}\right\}$, is included in the strictly lower contour set

$$\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}\left(I-A\right)x < \overline{p}b\right\} = \left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}Ax < \frac{1}{N}\right\}.$$

Thus, there is no RS in such a case. Second, even if wealth endowments are presumed to be equal, every nation is faced with the common set of feasible activities $\left\{x \in \mathbb{R}_{+}^{n} \mid \overline{p}\left(I-A\right)x = \overline{p}b\right\}$, which is not identical to the indifference surface $\left\{x \in \mathbb{R}_{+}^{n} \mid Lx = L\left(I-A\right)^{-1}b\right\}$, meaning that every nation ν would choose the same activity $x'^{\nu} = \arg\min_{x \in \mathbb{R}_{+}^{n};\ \overline{p}(I-A)x=\overline{p}b} Lx$ to minimize its own labor supply. Note that $Lx'^{\nu} < vb$ for any $\nu \in \mathcal{N}$, since while $\left(I-A\right)^{-1}b > \mathbf{0}$ holds, the solution of the program $\min_{x \in \mathbb{R}_{+}^{n};\ \overline{p}(I-A)x=\overline{p}b} Lx$ should be a boundary point of $\mathbb{R}_{+}^{n}$, which implies that $\left(I-A\right)x'^{\nu} \ngeqq b$ holds for any $\nu \in \mathcal{N}$. Thus, the aggregate net output does not coincide with Nb.

Let us consider a case that, given $\overline{p}$ and L are linearly independent, $\widetilde{p}\left(I-\left(1+r\right)A\right) - wL = \mathbf{0}$ for some $\widetilde{p} \in \triangle\left(\overline{\omega}\right)$, some $r \in [0, R)$, and some $w > 0$. Note that such a price vector $\left(\widetilde{p}, w, r\right)$ exists because of the productiveness and indecomposability of A. If $\widetilde{p}A$ and L are linearly dependent, then from Lemma 1, $\widetilde{p}$ is identical to the Frobenius eigenvector of A uniquely up to scale, meaning that $\overline{p}A$ and L are linearly dependent, which is a contradiction. Thus, $\widetilde{p}A$ and L are linearly independent. Then, according to Lemma 1, $\widetilde{p}$ cannot be proportional to v. This finding implies that $r > 0$ must hold, since $\widetilde{p}\left(I-A\right) - wL = \mathbf{0}$ implies $\frac{\widetilde{p}}{w} = v$, where w is determined to fulfill the gap between $\widetilde{p} \in \triangle\left(\overline{\omega}\right)$ and v. Then, since $\overline{p}$ and L are linearly independent, Theorem 3 shows that, given the suitable assignment of $\overline{\omega}$ among nations, $\left(\widetilde{p}, w, r\right)$ can constitute an inegalitarian RS, which implies that the desired result is obtained. □

For Theorem 3, the following three lemmas are proven.

Lemma A1: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A[I-A]^{-1}(Nb)$, for any price vector $p > \mathbf{0}$ associated with its unique equal profit rate $r \in [0, R)$, if there is no $\varsigma > 0$ such that $p(I-A) = \varsigma pA$, it follows that x^* is a solution to the following program:

$$\min_{Lx \in [0,N]} Lx, \text{ subject to } p(I-A)x \geqq pNb;\ pAx \leqq p\overline{\omega}. \tag{A.1}$$

Proof. Let $p \in \mathbb{R}^n_{++}$ be a price vector such that there exists $r \in [0, R)$ with $p = L(I-(1+r)A)^{-1}$, where $0 < \frac{1}{1+R} < 1$ is the unique Frobenius eigenvalue associated with A. Then, define $Y_1(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid p(I-A)x \geqq pNb\right\}$ and $Y_2(p) \equiv \left\{x \in \mathbb{R}^n_+ \mid pAx \leqq p\overline{\omega}\right\}$. Note that $p(I-A)x^* = pNb$ and $pAx^* = p\overline{\omega}$, thus $x^* \in Y_1(p) \cap Y_2(p)$.

If $r = 0$, then p is proportional to v, which implies that there is some $\varsigma > 0$ such that $p(I-A) = \varsigma v(I-A) = \varsigma L$, meaning that the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid p(I-A)x = pNb\right\}$ and the indifference surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = Nvb\right\}$ coincide. Therefore, if $r = 0$, x^* is an optimal solution, since any activity $x \in \mathbb{R}^n_+$ with $Lx < Lx^* = Nvb$ implies $p(I-A)x < pNb$.

Next, consider $r > 0$. In this case, if $x \in Y_1(p) \cap Y_2(p)$ is $pAx \leqq p\overline{\omega}$ and $p(I-A)x > pNb$, then for some small positive vector $\boldsymbol{\varepsilon} > \mathbf{0}$, $pA(x-\boldsymbol{\varepsilon}) < p\overline{\omega}$, $p(I-A)(x-\boldsymbol{\varepsilon}) \geqq pNb$, and $L(x-\boldsymbol{\varepsilon}) < Lx$ hold. Thus, x cannot be an optimal solution. Therefore, if $x \in Y_1(p) \cap Y_2(p)$ is an optimal solution to the program (A.1), then $p(I-A)x = pNb$ holds. Suppose that x^* is not an optimal solution to the program (A.1). Then, there should be another activity vector $x' \in Y_1(p) \cap Y_2(p)$ such that $Lx' < Lx^*$. Since $p(I-A)x' = pNb = p(I-A)x^*$, $pAx' \leqq p\overline{\omega} = pAx^*$, and $Lx' < Lx^*$, it follows that $[p(I-A) - rpA - L](x'-x^*) > 0$. However, since $p(I-A) - rpA - L = \mathbf{0}$ by definition, the aforementioned inequality is impossible. Thus, there is no such x', and x^* is a solution to the program (A.1). □

Lemma A2: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A[I-A]^{-1}(Nb)$, let a price vector $p > 0$ be associated with its unique equal interest rate $r \in [0, R)$ such that there is no $\varsigma > 0$ satisfying $p(I-A) = \varsigma pA$. Moreover, let $x^\nu \in \mathbb{R}^n_+$ be such that $p(I-A)x^\nu = pb$. Then, there exists $\omega^\nu \equiv Ax^\nu$ such that $x^\nu$is a solution to the following program:

$$\min_{Lx \in [0,1]} Lx, \text{ subject to } p(I-A)x \geqq pb;\ pAx \leqq p\omega^\nu. \tag{A.2}$$

Proof. Let $x^{\nu} \in \mathbb{R}_{+}^{n}$ be such that $p(I-A)x^{\nu} = pb$ and let $\omega^{\nu} \equiv Ax^{\nu}$. Let $Y_{1}^{\nu}(p) \equiv \left\{x \in \mathbb{R}_{+}^{n} \mid p(I-A)x \geqq p(I-A)x^{\nu}\right\}$ and $Y_{2}^{\nu}(p) \equiv \left\{x \in \mathbb{R}_{+}^{n} \mid pAx \leqq pAx^{\nu}\right\}$. Because of this supposition, the intersection $Y_{1}^{\nu}(p) \cap Y_{2}^{\nu}(p)$ has its interior set $int\left(Y_{1}^{\nu}(p) \cap Y_{2}^{\nu}(p)\right)$. Then, as shown in the proof of Lemma A1, for any $x' \in \left(Y_{1}^{\nu}(p) \cap Y_{2}^{\nu}(p)\right) \setminus \{x^{\nu}\}$, if x' is a solution to the program (**), then $p(I-A)x' = p(I-A)x^{\nu}$ and $pAx' \leqq pAx^{\nu}$. Suppose $Lx' < Lx^{\nu}$. Then, $[p(I-A) - rpA - L](x' - x^{\nu}) > 0$, which contradicts $p(I-A) - rpA - L = \mathbf{0}$. Thus, $Lx^{\nu} = Lx'$ holds, since x' is a solution to the program (**), which implies x^{ν} is a solution to the program (**). □

Given $\overline{\omega} = A[I-A]^{-1}(Nb)$, let

$$\overline{\Omega} \equiv \left\{ (\omega^{\nu})_{\nu \in N} \in \mathbb{R}_{+}^{nN} \mid \sum_{\nu \in \mathcal{N}} \omega^{\nu} = \overline{\omega} \ \& \ LA^{-1}\omega^{\nu} \in [0,1] \quad (\forall \nu \in \mathcal{N}) \right\}.$$

Lemma A3: Given an economy $\langle \mathcal{N}, (A, L, u), \overline{\omega} \rangle$ with $\overline{\omega} = A[I-A]^{-1}(Nb)$, let a price vector $p > \mathbf{0}$ be associated with its unique equal profit rate $r \in [0, R)$, such that there is no $\varsigma > 0$ satisfying $p(I-A) = \varsigma pA$. Then, there exists a suitable profile $(\omega^{\nu})_{\nu \in \mathcal{N}} \in \overline{\Omega}$ such that $LA^{-1}\omega^{\nu} \in [0,1]$ and $p(I-A)A^{-1}\omega^{\nu} = pb$ for any $\nu \in \mathcal{N}$, and $\left(p, (A^{-1}\omega^{\nu})_{\nu \in N}\right)$ constitutes an RS. In particular, if $A^{-1}\omega^{\nu'} \neq (I-A)^{-1}b$ for some $\nu' \in \mathcal{N}$, then $\left(p, (A^{-1}\omega^{\nu})_{\nu \in N}\right)$ constitutes an inegalitarian RS if and only if $r > 0$.

Proof. From the supposition about the price vector, $p = L(I-(1+r)A)^{-1}$ and there is no $\varsigma > 0$ such that $p(I-A) = \varsigma pA$ holds. Then, from Lemma A1, $\frac{x^*}{N}$ is a solution of $\min_{Lx \in [0,1]} Lx$ such that $p(I-A)x \geqq pb$ and $pAx \leqq \frac{p\overline{\omega}}{N}$. Take any profile $(\omega^{\nu})_{\nu \in N} \in \overline{\Omega}$ such that $LA^{-1}\omega^{\nu} \in [0,1]$ and $p(I-A)A^{-1}\omega^{\nu} = pb$ for any $\nu \in \mathcal{N}$ and $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ for some $\nu' \in \mathcal{N}$. Then, for each $\nu \in \mathcal{N}$, let $x^{\nu} \equiv A^{-1}\omega^{\nu}$. Lemma A2 implies that for each $\nu \in \mathcal{N}$, x^{ν} is a solution of $\min_{Lx \in [0,1]} Lx$ such that $p(I-A)x \geqq pb$ and $pAx \leqq p\omega^{\nu}$. Since $\sum_{\nu \in N} x^{\nu} = A^{-1}\overline{\omega} = x^*$, $\left(p, (x^{\nu})_{\nu \in N}\right)$ constitutes an RS.

Moreover, noting $\frac{x^*}{N} = (I-A)^{-1}b$, let us consider $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ for some $\nu' \in \mathcal{N}$. Then, owing to the setting of $p(I-A)A^{-1}\omega^{\nu'} = pb$, which is equivalent to $p(I-A)A^{-1}\omega^{\nu'} = p(I-A)\frac{x^*}{N}$, the property $A^{-1}\omega^{\nu'} \neq \frac{x^*}{N}$ implies that $p\omega^{\nu'} \neq \frac{p\overline{\omega}}{N}$ holds. Without loss of generality, let $p\omega^{\nu'} < \frac{p\overline{\omega}}{N}$. Then, there exists $p\omega^{\nu''} > \frac{p\overline{\omega}}{N}$ for another $\nu'' \in N$. Since $pb = rp\omega^{\nu} + Lx^{\nu}$

for each $\nu \in \mathcal{N}$, $Lx^{\nu'} > Lx^{\nu''}$ holds if and only if $r > 0$. This finding implies that $\left(p, (x^{\nu})_{\nu \in N}\right)$ is an inegalitarian RS if and only if $r > 0$. $\square$

Before the proof of Theorem 3, let us show that θ^r is well-defined for any $r \in (0, R)$. To show it, let $\theta^{r,b} \equiv \frac{p^r b}{p^r \overline{\omega}}$. Then, the system of equations is reduced to the form $p^r (I - A) x = p^r b = p^r A x$ with $Lx \in [0, 1]$. Let $\overline{X}\left(\theta^{r,b}\right)$ be the set of solutions satisfying $p^r (I - A) x = p^r b = p^r A x$. The set $\overline{X}\left(\theta^{r,b}\right)$ is non-empty and compact. Since $pb > 0$, $x = \mathbf{0} \notin \overline{X}\left(\theta^{r,b}\right)$. Note that for any $\theta < \theta^{r,b}$, no $x \in \mathbb{R}^n_+$ satisfies $p^r A x = \theta p^r \overline{\omega}$ and $p^r (I - A) x = p^r b$, since in such a case, the set of non-negative vectors, $x \in \mathbb{R}^n_+$, satisfying $p^r A x = \theta p^r \overline{\omega}$ is contained by the strictly lower contour set of the hyperplane defined by the supporting vector $p^r (I - A)$ at the point $(I - A)^{-1} b \in \mathbb{R}^n_+$. By contrast, for any $\theta \geqq \theta^{r,b}$, there is a non-empty set $\overline{X}(\theta) \subseteq \mathbb{R}^n_+$ such that for any $x \in \overline{X}(\theta)$, $p^r A x = \theta p^r \overline{\omega}$ and $p^r (I - A) x = p^r b$ hold.

Since each $\overline{X}(\theta)$ is compact, we can find the solution to the program $\min_{x \in \overline{X}(\theta)} Lx$ whenever $\overline{X}(\theta) \neq \varnothing$. Therefore, the program (*) can be reduced to the following form:

$$\min_{\theta \in \left[\theta^{r,b}, 1\right]} \theta, \text{ subject to } \min_{x \in \overline{X}(\theta)} Lx \leqq 1. \quad (**)$$

Since $\min_{x \in \overline{X}(\theta)} Lx$ is decreasing with respect to $\theta \in \left[\theta^{r,b}, 1\right]$ and $\min_{x \in \overline{X}\left(\frac{1}{N}\right)} Lx = vb < 1$, there exists $\theta^r \in \left[\theta^{r,b}, \frac{1}{N}\right]$, which is the solution to the program (**).

Now, we can prove Theorem 3:

Proof of ***Theorem 3***. Given the technology (A, L), let us consider $\overline{p}\,(I - (1 + R) A) = \mathbf{0}$, where $\frac{1}{1+R}$ is the unique Frobenius eigenvalue of A and $\overline{p} > \mathbf{0}$ is its associated eigenvector unique up to scale. Therefore, let us suppose that $\overline{p} \in \triangle(\overline{\omega})$. By definition, the row vectors $\overline{p}$ and $\overline{p}A$ are linearly dependent. From this supposition, it follows that $\overline{p}$ and L are linearly independent. Then, $\overline{p}A$ and L are linearly independent. From Lemma 1, $\overline{p}$ cannot be a positive price vector associated with a non-negative positive interest rate $r \in [0, R)$. In addition, because of the uniqueness of the Frobenius eigenvector of indecomposable A, no positive price vector $p \in \triangle(\overline{\omega})$ associated with an equal interest rate $r \in [0, R)$ can be the Frobenius eigenvector, since the Frobenius eigenvector $\overline{p} > \mathbf{0}$ cannot be associated with a non-negative interest rate $r \in [0, R)$.

Therefore, according to Lemma 1, for any such p, pA and L are linearly independent.

Let us consider $r = 0$, meaning that $\widetilde{p}(I-A) - wL = \mathbf{0}$ for some $\widetilde{p} \in \triangle(\overline{\omega})$. Then, as before, $\widetilde{p}$ is proportional to the labor value vector v, $\frac{\widetilde{p}}{\widetilde{w}} = v$, where $\widetilde{w}$ is determined to fulfill the gap between $\widetilde{p} \in \triangle(\overline{\omega})$ and v. Hence, there exists an RS even under unequal initial endowments of financial capital and such an RS is always egalitarian. Indeed, in this case, since $\widetilde{p}(I-A)$ and L are linearly dependent according to $\widetilde{p}(I-A) - wL = \mathbf{0}$, the hyperplane $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I-A)x = \widetilde{p}b\right\}$ and the indifference surface $\left\{x \in \mathbb{R}^n_+ \mid Lx = vb\right\}$ coincide. Hence, for each nation $\nu \in \mathcal{N}$, the intersection of $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I-A)x = \widetilde{p}b\right\}$ and $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}Ax \leqq W^\nu\right\}$ constitutes the set of optimal activities at the price $\widetilde{p} \in \triangle(\overline{\omega})$. Therefore, for any $(x^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$ with $(I-A)\left(\sum_{\nu\in\mathcal{N}} x^\nu\right) = Nb$, $\left(\widetilde{p}, (x^\nu)_{\nu\in\mathcal{N}}\right)$ can constitute an RS if and only if there exists an assignment $(W^\nu)_{\nu\in\mathcal{N}}$ such that

$$x^\nu \in \left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I-A)x = \widetilde{p}b\right\} \cap \left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}Ax \leqq W^\nu\right\}.$$

This finding also implies that $Lx^\nu = vb < 1$ for any $\nu \in \mathcal{N}$, from the identity of $\left\{x \in \mathbb{R}^n_+ \mid \widetilde{p}(I-A)x = \widetilde{p}b\right\}$ with $\left\{x \in \mathbb{R}^n_+ \mid Lx = vb\right\}$, regardless of whether $(W^\nu)_{\nu\in\mathcal{N}}$ is unequal. For instance, if $(W^\nu)_{\nu\in\mathcal{N}}$ is equalized, then $x^\nu = (I-A)^{-1}b$ is the unique optimal solution for any agent $\nu \in \mathcal{N}$. If $(W^\nu)_{\nu\in\mathcal{N}} \in \triangle(W)$ represents an unequal distribution, but it meets the property that $\min_{\nu\in\mathcal{N}} W^\nu \geqq \min_{\omega\leqq\overline{\omega};\ LA^{-1}\omega=vb} \widetilde{p}\omega$, then there exists a suitable assignment $(\omega^\nu)_{\nu\in\mathcal{N}}$ of $\overline{\omega}$ such that $\widetilde{p}\omega^\nu = W^\nu$ for any $\nu \in \mathcal{N}$ and for some $(x^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$, $x^\nu = A^{-1}\omega^\nu$. By definition, $\widetilde{p}Ax^\nu = \widetilde{p}\omega^\nu$ and $Lx^\nu = vb$ for all $\nu \in \mathcal{N}$, which also implies $\widetilde{p}(I-A)x^\nu = \widetilde{p}b$ for all $\nu \in \mathcal{N}$. Thus, this RS is egalitarian, although its initial distribution of financial capital is unequal. In such an equilibrium, international division of labor is generated by the differences in the capital-labor ratios among nations. Because every nation supplies the same amount of labor vb, $W^\nu > W^{\nu'}$ implies that ν is specialized to a more capital-intensive production activity than ν' is.

Let us consider $r \in (0, R)$, which allows us to find a unique price vector $p^r = L(I-(1+r)A)^{-1} > \mathbf{0}$ and p^rA and L are linearly independent, according to Lemma 1. By definition, $(W^\nu)_{\nu\in\mathcal{N}} \in \triangle_{\underline{\theta}}(W)$ implies that there exists $r^* \in (0, R)$ such that $\min_{\nu\in\mathcal{N}} W^\nu \geqq \theta^{r^*}$ and for some $p^{r^*} = L(I-(1+r^*)A)^{-1}$, there exists $x(r^*) \in \mathbb{R}^n_{++}$ such that $p^{r^*}Ax(r^*) = \theta^{r^*}p^{r^*}\overline{\omega}$, $p^{r^*}(I-A)x(r^*) = p^{r^*}b$, and $Lx(r^*) \in [0,1]$. Then, there exist $p^* \equiv \frac{1}{L(I-(1+r^*)A)^{-1}\overline{\omega}} L(I-(1+r^*)A)^{-1} \in \triangle(\overline{\omega})$ and $(\omega^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$ such that $p^*\omega^\nu = W^\nu$ for any $\nu \in \mathcal{N}$. Let us define $w^* > 0$ to fulfill the gap between p^* and p^{r^*} as $p^* = w^* p^{r^*}$.

Since $\min_{\nu \in \mathcal{N}} W^\nu > \underline{\theta}$, there exists $(x^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_{++}$ such that for each $\nu \in \mathcal{N}$, $p^* A x^{*\nu} = p^* \omega^\nu$, $p^* (I - A) x^{*\nu} = p^* b$, and $L x^{*\nu} \leqq 1$. From Lemma A2, for a profile $(\omega^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_{++}$ with $\omega^{*\nu} \equiv A x^{*\nu}$ for each $\nu \in \mathcal{N}$, $x^{*\nu}$ is a solution to the program (**). Then, since $p^*\omega^\nu = p^*\omega^{*\nu}$ for each $\nu \in \mathcal{N}$, $x^{*\nu}$ is also a solution to the following optimization program:

$$\min_{Lx \in [0,1]} Lx, \text{ subject to } p^* (I - A) x \geqq p^* b;\ p^* A x \leqq p^* \omega^\nu.$$

Since $x^{*\nu} = A^{-1}\omega^{*\nu}$ for each $\nu \in \mathcal{N}$, Lemma A3 applies, meaning that $\langle p^*; (w^*, r^*), (x^{*\nu})_{\nu \in \mathcal{N}} \rangle$ is an inegalitarian RS if and only if $(W^\nu)_{\nu \in \mathcal{N}}$ is an unequal distribution. □

Proof of ***Theorem 4***. Note that for any $\overline{\omega} \geqq A [I - A]^{-1} (Nb)$, the social production activity $\sum_{\nu \in \mathcal{N}} x^{*\nu}$ of any RS is equal to $x^* = [I - A]^{-1} (Nb)$ and its associated equilibrium price vector p^* meets $p^* - p^* A > \mathbf{0}$, according to Proposition 1.

For the unique Frobenius eigenvector $\overline{p} > \mathbf{0}$ of A, consider $\overline{p}$ and L to be linearly dependent. Then, since $\overline{p}A$ and L are also linearly dependent, Lemma 1 implies that any RS price vector is characterized by labor value pricing. Then, for any $(\omega^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_{+}$ such that $v\omega^\nu \geqq vA\frac{x^*}{N}$ for any $\nu \in \mathcal{N}$, $\langle \overline{p}; (w, r), \left(\frac{x^*}{N}, \ldots, \frac{x^*}{N}\right) \rangle$ with $r \in [0, R)$ and $w > 0$ such that $\overline{p} = (1 + r)\overline{p}A + wL$ constitutes an egalitarian RS. In this case, there should be a nation ν having $\overline{p}\omega^\nu > \overline{p}A\frac{x^*}{N}$ according to $\overline{\omega} \geq A [I - A]^{-1} (Nb)$ and $\overline{p} > \mathbf{0}$. However, $\frac{x^*}{N}$ is still an optimal activity for this agent.

Consider the case that $\overline{p}$ and L are linearly independent. In this case, we cannot apply Lemma 1 and Theorem 1, since an RS $\langle p^*; (w^{\nu *}, r^{\nu *})_{\nu \in \mathcal{N}}, (x^{*\nu})_{\nu \in \mathcal{N}} \rangle$ under $\overline{\omega} \geq A [I - A]^{-1} (Nb)$ may not involve incomplete specialization. According to the definition of (MP_t^ν), we have $p^* (I - A) x^* = r^* p^* A x^* + w^* L x^*$, where $r^* \equiv \frac{\sum_{\nu \in \mathcal{N}} r^{\nu *} p^* A x^{*\nu}}{p^* A x^*}$ and $w^* \equiv \frac{\sum_{\nu \in \mathcal{N}} w^{\nu *} L x^{*\nu}}{L x^*}$. Therefore, if $p^* A$ and L are linearly dependent, then Lemma 1 implies that $\overline{p}A$ and L are linearly dependent, which is a contradiction. Thus, let us focus on the case that $p^* A$ and L are linearly independent. Then, whenever $r^* > 0$, $p^* (I - A)$ and L are linearly independent.

Suppose that $r^* > 0$. Note that x^* is a solution to the following program:

$$\min_{Lx \in [0,N]} Lx, \text{ subject to } p^* (I - A) x = p^* Nb; p^* Ax \leqq p^* Ax^*. \qquad \text{(A.1')}$$

Then, there exists $x' \geq \mathbf{0}$ such that $p^* (I - A) x' = p^* Nb$, $Lx' < Lx^*$, and $p^* Ax' > p^* Ax^*$ because $p^* (I - A)$ and L are linearly independent, and $x^* = [I - A]^{-1} (Nb) > \mathbf{0}$. In fact, suppose that for any $x \in \mathbb{R}^n_+ \setminus \{x^*\}$ such that $p^* (I - A) x = p^* Nb$, if $p^* Ax > p^* Ax^*$, then $Lx \geqq Lx^*$. This finding implies that for any $x \in \mathbb{R}^n_+ \setminus \{x^*\}$ such that $p^* (I - A) x = p^* Nb$, if $Lx < Lx^*$, then $p^* Ax \leqq p^* Ax^*$. Thus, if there exists $x' \in \mathbb{R}^n_+ \setminus \{x^*\}$ such that $Lx' < Lx^*$ and $p^* (I - A) x' = p^* Nb$, then $p^* Ax' \leqq p^* Ax^*$, which contradicts the fact that x^* is a solution to the program (A.1'). Therefore, for any $x \in \mathbb{R}^n_+ \setminus \{x^*\}$ such that $p^* (I - A) x = p^* Nb$, $Lx \geqq Lx^*$ holds. However, since $p^* (I - A)$ and L are linearly independent, this finding implies that for any $x \in \mathbb{R}^n_+ \setminus \{x^*\}$ such that $p^* (I - A) x = p^* Nb$, $Lx > Lx^*$ holds. Such a situation is possible only when x^* is in the boundary of $\mathbb{R}^n_+$. However, since $x^* > \mathbf{0}$, this is a contradiction. Thus, we must conclude that there exists $x' \geq \mathbf{0}$ such that $p^* (I - A) x' = p^* Nb$, $Lx' < Lx^*$, and $p^* Ax' > p^* Ax^*$. Then, define a convex combination $x^{*\prime} \equiv \epsilon x' + (1 - \epsilon) x^*$ for sufficiently small positive ϵ. By definition, $p^* (I - A) x^{*\prime} = p^* Nb$, $Lx^{*\prime} < Lx^*$, and $p^* Ax^{*\prime} > p^* Ax^*$.

Since $\overline{\omega} \geq A [I - A]^{-1} (Nb)$ implies $p^* \overline{\omega} > p^* Ax^*$, we can have $p^* Ax^{*\prime} \leqq p^* \overline{\omega}$ for a sufficiently small positive ϵ. This finding implies that given a suitable assignment of $x^{*\prime}$ among the members of $\mathcal{N}$, there should be at least one nation $\nu \in \mathcal{N}$ such that $p^* (I - A) x^{*\prime\nu} = p^* b$, $p^* Ax^{*\prime\nu} \leqq p^* \omega^\nu$, and $Lx^{*\prime\nu} < Lx^{*\nu}$. However, this is a contradiction, since $\left\langle p^*; (w^{\nu *}, r^{\nu *})_{\nu \in \mathcal{N}}, (x^{*\nu})_{\nu \in \mathcal{N}} \right\rangle$ is an RS. Therefore, $r^* \not> 0$ for an RS under $\overline{\omega} \geq A [I - A]^{-1} (Nb)$, meaning that $r^* = 0$ holds. Then, $p^* = w^* v$. As shown above, such an RS is egalitarian even for $\overline{\omega} \geq A [I - A]^{-1} (Nb)$. □

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