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Tatsuyoshi Saijo

Research Center for Future Design Engineering, Kochi University of Technology

Institute of Economic Research, Hitotsubashi University

Center for Environmental Innovation Design for Sustainability, Osaka University

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School of Economics and Management
Research Center for Social Design Engineering
Kochi University of Technology

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Tatsuyoshi Saijo*
Research Center for Social Design Engineering, Kochi University of Technology
Institute of Economic Research, Hitotsubashi University, and
Center for Environmental Innovation Design for Sustainability, Osaka University

Abstract

I consider systems with voluntary contribution mechanisms (VCM) in a quasilinear environment, showing that they are *not asymptotically stable* under simultaneous difference equations, and are *structurally unstable* under simultaneous differential equations. However, a VCM system is stable in a Cobb–Douglas (CD) environment under simultaneous differential equations; it is also stable under simultaneous difference equations given conditions on the slope of best response functions and the number of players. Most of the parameters investigated in nonlinear experiments on the VCM so far have been on the boundaries between stability and instability, and there has been hardly any work on instability. Therefore, research on the private provision of public goods faces a new challenge in addition to free riding.

JEL classification codes: C62, C72, C92, H41

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1. Introduction

Numerous papers using human subjects have been devoted to research on the voluntary contribution mechanism (VCM) in order to understand the public good provision problem.[1] Researchers chose utility functions at their discretion, without investigating what types of utility functions people actually have. Almost all experimental economists chose linear utility functions such as $u(x_i, y) = x_i + by$ where x_i is a private good and y a public good. Under this linear environment, they found human behaviors such as altruism, equity, warm-glow, and spite. They also found conditions for the efficient provision of public goods—conditions that built on these aspects of human nature—in addition to devices including punishment, reward, and communication. However, what if utility functions are not linear?

A typical public good is a network of roads. This public good is related to commuting from one's home to the office, to shops, to venues for leisure, and so on. For example, work performance might be affected by commuting time. Some expansion of the network far away from one's home might not affect one's daily life. All academic personnel belong to some committees that provide public goods such as evaluation systems for faculty members, employment of new members, construction of new programs, and so on. All members devote some time to these committees, but none devote all of their time to them. That is, private and public goods are related to each other, and their relationship in our utility functions is not linear.

A few experimental economists started investigating the properties of the VCM with *nonlinear* utility functions. Innovators such as Isaac, McCue, and Plott (1985), Isaac and Walker (1991), Sefton and Steinberg (1996), Isaac and Walker (1998), and Laury, Walker, and Williams (1999) used quasilinear utility functions that are linear with respect to player *i*'s private good consumption x_i and nonlinear with respect to a public good y, that is, $u(x_i, y) = x_i + t(y)$.[2] I call these utility functions QL1. The rationality behind this formulation is that the private good is money and hence its marginal utility is constant, but the marginal utility of the public good decreases so that the $t(y)$ part is nonlinear. Andreoni (1993), Chan, Mestelman, Moir, and Muller (1996), Cason, Saijo, and Yamato (2002) and Sutter and Weck-Hannemann (2004) started using Cobb–Douglas utility functions. I call these utility functions CD. A basic reason for using this type of utility function is to avoid the multiplicity of Nash equilibria in the VCM obtained with QL1 utility functions. That is, the Nash equilibrium is unique with CD utility functions. Another type of utility function, nonlinear with respect to x_i and linear with respect to y, that is,

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[1] See Ledyard (1995) and Chaudhuri (2011) for surveys on experiments. Bergstrom, Blum, and Varian (1986) discuss the basic theoretical properties of the VCM.

[2] Isaac, McCue, and Plott (1985) were the first pioneering experimentalists to use QL1 utility functions in VCM experiments; Isaac and Walker (1991) also used QL1. In their choice of parameters, no contribution is the dominant strategy.

$u(x_i, y) = h(x_i) + y$, has been used by Sefton and Steinberg (1996), Keser (1996), Falkinger, Fehr, Gächter, and Winter-Ebmer (2000), van Dijk, Sonnemans, and van Winden (2002), Uler (2011), Maurice, Rouaix, and Willinger (2013) and Cason and Gangadharan (2014) among others.[3] I call these utility functions QL2. This utility function makes the peculiar assumption that the marginal utility of the private good decreases, while that of the public good is constant. An advantage of this utility function is that players' dominant strategies are similar to their dominant strategies in a linear environment.

In the VCM, each player has some endowment of a private good and must decide on a contribution out of this endowment. The level of the public good is the sum of the contributions. If b, termed the marginal per capita return, is less than one in the linear utility function, each player chooses zero contribution, regardless of the other players' contributions. That is, no contribution is the *rational* choice, or the dominant strategy. On the other hand, if utility functions are nonlinear, the Nash equilibrium is usually located in the interior of the consumption set, and players with QL1 or CD utility functions do not have any dominant strategy.

If players do not have dominant strategies, but have Nash equilibrium strategies, it becomes important to consider the stability of the system, as researchers in other fields such as industrial organization do. Since players have dominant strategies in the QL2 environment, I focus upon QL1 and CD environments for my stability analysis.

First, I consider a system of simultaneous *difference* equations using best response dynamics assuming that every player has the same utility function and identical endowments. Surprisingly, any VCM system in the QL1 environment is *unstable* regardless of the number of players. If the environment is CD, the stability condition is $|a(n-1)| < 1$ where a is the slope of the best response function and n is the number of players. Second, let us consider the system of simultaneous *differential* equations using best response dynamics. Although the VCM system with QL1 is stable, it is on the bifurcation point. That is, a slight parameter change alters the properties of the Nash equilibrium, and affects stability. In this sense, the system is *structurally unstable*. On the other hand, the system is stable in the CD environment.

Since most of the experiments mentioned above were repeated for many periods, the "difference" model is more appropriate than the "differential" model. Most experiments, excluding QL2 experiments, employed unstable systems in the form of difference equations, but the analytical focus was based upon the Nash equilibrium. That is, unstable parts of experimental results might have been overlooked. Therefore, research on the private provision of public goods faces a new challenge in addition to free riding.

The organization of the rest of the paper is as follows. I consider stability properties under

[3] There are many other experimental papers in this category and the number has been growing.

difference equations in section 2 and under differential equations in section 3. Section 4 summarizes previous experimental results with nonlinear utility functions. Section 5 suggests avenues for further research.

2. Stability properties: difference equations

Let x be a private good and y be a public good. The production function of the public good is $y = f(x) = x$. That is, for example, one hour of labor input produces one meter of road. Player i has endowment w and must decide to divide w into i's own consumption of the private good x_i and the contribution s_i to the public good. That is, $y = \sum_1^n s_j$ where n is the number of players and n is at least two. This system is called the voluntary contribution mechanism (VCM). I assume that all players have the same endowment and the same utility function $u(x_i, y)$. Then, player i faces the following problem.

$Max\ u(x_i, s_i + s_{-i})$ subject to $w = x_i + s_i$

where $s_{-i} = \sum_{j \neq i} s_j$. Let $u(w - s_i, s_i + s_{-i}) = v(s_i, s_{-i})$. A list of contributions $\hat{s} = (\hat{s}_1, ..., \hat{s}_n)$ is a *Nash equilibrium* if for all i $v(\hat{s}_i, \hat{s}_{-i}) \geq v(s_i, \hat{s}_{-i})$ for all $s_i \in [0, w]$. Define the best response function as

$r(s_{-i}) = \arg\max_{s_i} \{v(s_i, s_{-i}) \mid s_i \in [0, w]\}$.[4]

Since v and w are the same for all players, all players have the same best response function.

So far, experimental economists have been using three types of nonlinear utility functions in public good provision experiments:

(QL1) $u(x_i, y) = x_i + t(y)$;
(CD) $u(x_i, y) = x_i^{\alpha} y^{1-\alpha}$, $\alpha \in (0,1)$; and
(QL2) $u(x_i, y) = h(x_i) + y$.

Once we consider x as money and assume that the marginal utility of money is constant and that the marginal utility of the public good is decreasing, it is quite natural to consider QL1 utility functions, in the spirit of Isaac, McCue, and Plott (1985), Isaac and Walker (1991), Sefton and Steinberg (1996), Isaac and Walker (1998), Laury, Walker, and Williams (1999), and others. Andreoni (1993), Chan, Mestelman, Moir, and Muller (1996), Cason, Saijo, and Yamato (2002), Sutter and Weck-Hannemann (2004), and others used CD utility functions. Sefton and Steinberg (1996), Keser (1996), Falkinger, Fehr, Gächter, and Winter-Ebmer (2000), van Dijk, Sonnemans, and van Winden (2002), Uler (2011), Maurice, Rouaix, and Willinger (2013), Cason and Gangadharan (2014) and others adopted QL2 utility functions in which the public good is in the linear form and the private good is in the nonlinear form. All three types have the common

[4] If u is strictly quasi-concave, the maximizer is unique and r is continuous by Berge's maximum theorem.

property that the best response functions are linear. In order to see this property, consider the first order condition for the maximization problem, i.e., $-\partial u/\partial x_i + \partial u/\partial y = 0$. Then, totally differentiating both sides of the condition, we have the slope of the best response function:

$$\frac{dr}{ds_{-i}} = -\frac{\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial^2 u}{\partial x_i \partial y}}{\dfrac{\partial^2 u}{\partial x_i^2} - 2\dfrac{\partial^2 u}{\partial x_i \partial y} + \dfrac{\partial^2 u}{\partial y^2}}.$$

If the utility function is QL1, since $\partial^2 u/\partial x_i^2 = 0$ and $\partial^2 u/\partial x_i \partial y = 0$, $dr/ds_{-i} = -1$. In the same way as for QL1, we can obtain the slopes of the best response functions for CD and QL2.

(1-1) $\dfrac{dr}{ds_{-i}} = -1$, (1-2) $\dfrac{dr}{ds_{-i}} = -\alpha$, and (1-3) $\dfrac{dr}{ds_{-i}} = 0$,

where (1-1) is for QL1, (1-2) for CD, and (1-3) for QL2. Since all slopes are constant, the best response functions are piecewise linear, putting the best response zero on the negative part (*j-2w* in Figure 1, where the horizontal axis represents the sum of the other players' contributions and the vertical axis represents the contribution of player *i*'s response to s_{-i}). Then the best response functions are

(2-1) $s_i = \max\{-s_{-i} + t'^{-1}(1), 0\}$, [5]

(2-2) $s_i = \max\{-\alpha s_{-i} + (1-\alpha)w, 0\}$, [6] and

(2-3) $s_i = \max\{w - h'^{-1}(1), 0\}$, [7]

where (2-1) is for QL1, (2-2) for CD, and (2-3) for QL2 and where t'^{-1} and h'^{-1} are the inverse functions of $t' = dt/dy$ and $h' = dh/dy$ respectively.

At time *t*, let player *i*'s choice of contribution be s_i^t. Although there might be many ways to associate s_i^t with previous information such as $(s_i^{t-1}, s_{-i}^{t-1}), (s_i^{t-2}, s_{-i}^{t-2})$ and so on, I simply assume that player *i* chooses $r(s_{-i}^t)$ at time *t*+1, where *t* = 1, 2… That is, I assume that every player chooses the best response to the sum of strategies chosen by the other players in the immediately preceding period. Then the system of simultaneous difference equations

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5 Since $v(s_i, s_{-i}) = w - s_i + t(s_i + s_{-i})$, the first order condition is $\partial v/\partial s_i = -1 + t'(s_i + s_{-i}) = 0$. That is, $t'^{-1}(1) = s_i + s_{-i}$.

6 Since $v(s_i, s_{-i}) = (w - s_i)^{\alpha}(s_i + s_{-i})^{1-\alpha}$, the first order condition is $\partial v/\partial s_i = -\alpha((s_i + s_{-i})/(w - s_i))^{1-\alpha}$ $+(1-\alpha)((w - s_i)/(s_i + s_{-i}))^{\alpha} = 0$. Then $(s_i + s_{-i})/(w - s_i) = (1-\alpha)/\alpha$, and hence $s_i = -\alpha s_{-i} + (1-\alpha)w$.

7 Since $v(s_i, s_{-i}) = h(w - s_i) + (s_i + s_{-i})$, the first order condition is $\partial v/\partial s_i = -h'(w - s_i) + 1 = 0$. That is, $h'^{-1}(1) = w - s_i$.

$s_i^{t+1} = r(s_{-i}^t)$ (i = 1,2,⋯,n) is *locally stable* at the Nash equilibrium $\hat{s}$ if the system $s_i^{t+1} = r'(\hat{s}_{-i})s_{-i}^t + k_i$ (i = 1,2,⋯,n) is stable, where $k_i = \hat{s}_i - r'(\hat{s}_{-i})\hat{s}_{-i}$. That is, the system is a linear approximation to the original system at the Nash equilibrium. Since the best response function is the same for all players and is continuous, we have the symmetric Nash equilibrium. Let a be the slope of the best response function at the Nash equilibrium. Then, Saijo and Kobayashi (2014) proved the following property.[8]

Property 1. *The system $s_i^{t+1} = r(s_{-i}^t)$ (i = 1, 2,⋯, n) is locally asymptotically stable if and only if $|a(n-1)| < 1$.*[9]

Although the system $s_i^{t+1} = r'(\hat{s}_{-i})s_{-i}^t + k_i$ (i = 1,2,⋯,n) is the linear approximation of the system $s_i^{t+1} = r(s_{-i}^t)$ (i = 1,2,⋯,n) at the Nash equilibrium $\hat{s}$, both systems are the same if the best response functions are linear. If a utility function is QL1, the system is not stable since a = -1 and $|a(n-1)| = (n-1) \geq 1$ since $n \geq 2$. That is,

Proposition 1. *If the utility function is QL1, the system $s_i^{t+1} = r(s_{-i}^t)$ (i = 1, 2,⋯, n) is not asymptotically stable.*

Notice that since $s_i + s_{-i} = t'^{-1}(1)$ for all i excluding negative contributions, the set of Nash equilibria is $\{(s_1, s_2, ..., s_n) \mid \sum s_j = t'^{-1}(1), s_j \geq 0\}$ where $t'^{-1}(1)$ is the intercept g in Figure 1. Laury and Holt (2008) and Sefton and Steinberg (1996) pointed out that the relatively large standard deviation of each player's contributions might come from multiple Nash equilibria.

Consider Figure 1, drawn for QL1. All players have the same endowment w and g-j-$2w$ is the best response function. Consider the case of two players, that is, the left half of the box, 0-w-v-w. Then $|a(n-1)| = 1(2-1) = 1$. The slope of line $0v$ is 1; hence, the symmetric Nash equilibrium must be at the intersection of lines $0v$ and $r(s_{-i})$, which is e. Suppose, for example, that both players choose a. If this is the case, the best response is b. Then, since both choose b, the next best response is c, and the next response is b. That is, the system cycles between b and c. As an example, let g be 12 and let $(s_1^1, s_2^1) = (k, m)$, $k + m \neq 12$, and $k, m \geq 0$ and consider the case where each chooses different contribution: let $s^1 = (9,7)$. Then $s^2 = (5,3)$ and $s^3 = (9,7)$. That is, the system gets into the cycle between (9, 7) and (5, 3). Since the numbers in (9, 7) and (5, 3) do not hit the boundaries 0 and 12, I call the cycle an *interior* cycle.[10] If the sum of contributions is 12, i.e., $s^1 = (12-k, k)$ and $0 \leq k \leq 12$, the system stays at (12-k, k) which is a Nash equilibrium. Among the Nash equilibria, (6, 6) is the symmetric Nash equilibrium.

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[8] I ignore the negative part of the best response functions for mathematical simplicity.

[9] An intuitive interpretation of asymptotical stability of differential equations is given by Hirsch and Smale (1974). "An equilibrium $\bar{x}$ is *stable* if all nearby solutions stay nearby. It is *asymptotically stable* if all nearby solutions not only stay nearby, but also tend to $\bar{x}$" (p.180).

[10] "A fixed point is called *neutral* or *indifferent* if it is neither attracting nor repelling" (p.331) of Hirsch, Smale, and Devaney (2013). In this sense, the system with QL1 is *neutrally stable* if n = 2.

Consider the case of three players, that is, the entire box, 0-2*w*-*z*-*w*. Then $|a(n-1)| = 2 > 1$. The slope of line *0z* is 1/2 and hence the symmetric Nash equilibrium must be at the intersection of lines *0z* and $r(s_{-i})$, which is *e′*. Suppose that every player's input is the same at *a*/2. If this is the case, the best response is *b*. Then, the next best response should be *c′* and the best response to *c′* is *g*. That is, the best response to *g* is *d* and thereafter, the system cycles between *d* and *g*. As an example, let *g* be 12 and suppose that the sum of contributions is not 12 and let $s^1 = (s_1^1, s_2^1, s_3^1) = (9,7,2)$. Then $s^2 = (3,1,0)$, $s^3 = (11,9,8)$, $s^4 = (0,0,0)$, and $s^5 = (12,12,12)$. Then the system gets into the cycle between (0, 0, 0) and (12, 12, 12). Since both 12 and 0 are the boundaries of the best response function, I call this cycle a *boundary* cycle. If the sum of contributions is 12, the system stays at s^1 which is a Nash equilibrium. Among the Nash equilibria, (4, 4, 4) is a symmetric Nash equilibrium.

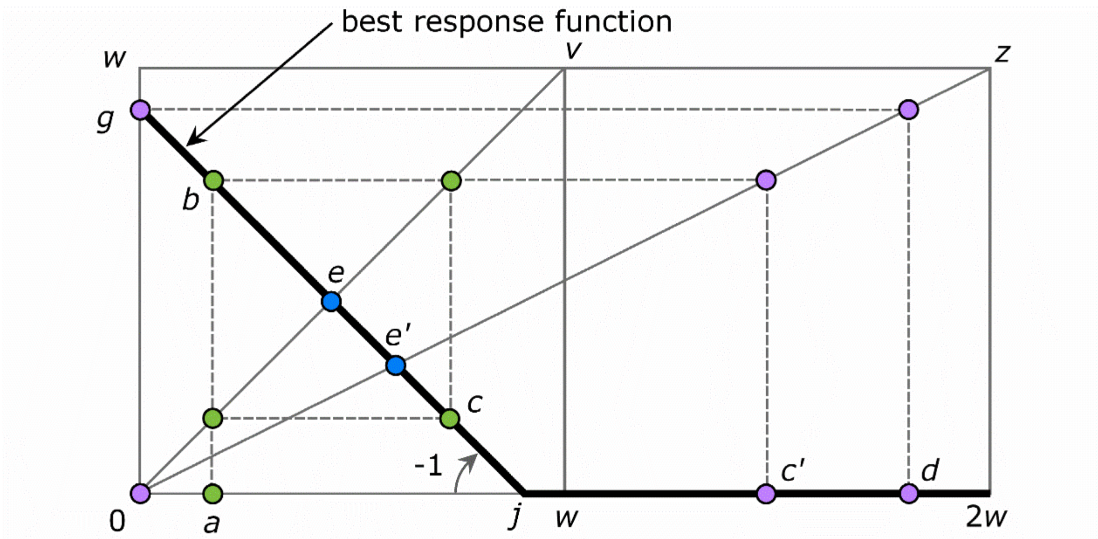

Figure 1. The best response function when the utility function is QL1.

The difference between the two-player and the three-player cases is in the nature of the cycles, i.e., either interior or boundary. If every player contributes $0 < s_i^1 < 12$ and the sum of the contributions is not 12, the system gets into the interior cycle immediately when *n* = 2, but hits the boundaries (0, 0, 0) and (12, 12, 12) within a few periods when *n* = 3. In this sense, the degree of instability is relatively more severe for the three-player case.

Consider the case with CD utility functions. Unlike with QL1 utility functions, the Nash equilibrium with CD functions is unique as long as every player has the same utility function and the same endowment. Then using Property 1, if *n* = 2, $|\alpha(n-1)| < 1$ since $0 < \alpha < 1$. Although it is trivial, there is a parameter range in which the system is stable.

Proposition 2. *Suppose that the utility function is CD.*

(*i*) *If n = 2, the system* $s_i^{t+1} = r(s_{-i}^t)$ *(i = 1, 2,⋯, n) is asymptotically stable; and*

(*ii*) *If* $n \geq 3$ *and* $\alpha < 1/(n-1)$, *the system is asymptotically stable.*

As in Chan, Mestelman, Moir, and Muller (1996), let n = 3, α = ½, and w = 20. Then $|\alpha(n-1)| = 1$ and $s_i = \max\{-(1/2)s_{-i} + 10, 0\}$. The intersection between $s_i = -(1/2)s_{-i} + 10$ and $s_i = (1/2)s_{-i}$ is (10, 5) and hence (5, 5, 5) is the unique Nash equilibrium. As an example, let $s^1 = (4,4,4)$. Then $s^2 = (6,6,6)$ and $s^3 = (4,4,4)$. That is, the system gets into the interior cycle between (4, 4, 4) and (6, 6, 6).[11] When n = 3, interior cycles would never happen if the utility function is QL1.

On the other hand, since the best response for QL2 is the horizontal line and since this is the dominant strategy, the system is always stable.

3. Stability properties: differential equations

I formulated the dynamic model of the VCM using a system of simultaneous difference equations in the previous section, and will consider it using a continuous adjustment process in this section. Since the best response functions using QL1 and CD are linear excluding the negative part, I will consider the system of simultaneous linear differential equations expressed by

(3) $\dot{s}_i(t) = a \cdot s_{-i}(t) + k_i - s_i(t)\ (i = 1, 2, \cdots, n)$.

Since all players have identical utility functions and endowments, $k_i = k$ for all i, where $k = t'^{-1}(1)$ for QL1 and $k = w(1-\alpha)$ for CD. Increases or decreases in player i's contribution is governed by the difference between the best response to the sum of other players' contributions at time t $(a \cdot s_{-i}(t) + k)$ and player i's contribution at time $t\,(s_i(t))$. If the difference, i.e., $\{a \cdot s_{-i}(t) + k\} - s_i(t)$, is positive (negative), player i's contribution increases (decreases) depending upon the magnitude of the difference. Using matrix notation, (3) can be expressed by

$$\dot{\mathbf{s}}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{k} - \mathbf{s}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{s}(t) + \mathbf{k}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{A} = a\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$, $\mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{bmatrix}$ and $\mathbf{k} = \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix}$.

Then Saijo and Kobayashi (2014) proved the following property.

Property 2. *If a > -1, the system* $\dot{\mathbf{s}}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{s}(t) + \mathbf{k}$ *is asymptotically stable at the Nash equilibrium.*

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[11] This shows that the system is neutrally stable.

As a direct corollary, we have the following proposition.

Proposition 3. *If the utility function is CD, the system* $\dot{\mathbf{s}}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{s}(t) + \mathbf{k}$ *is asymptotically stable at the Nash equilibrium.*

If the utility function is *CD*, $a = -\alpha$ and hence the stability condition in Property 2 is automatically satisfied since $\alpha \in (0,1)$. On the other hand, when the utility function is QL1, a = -1, and hence we cannot apply Property 2 to the QL1 case. The stability property is governed by the eigenvalues of $\mathbf{A} - \mathbf{I}$ and they are $((n-1)a-1, \underbrace{-a-1, -a-1, \dots, -a-1}_{n-1})$.[12] For example, if a = -1 and n = 2, they are -2 and 0. As an example, let k = 25. The following is a solution of the system of differential equations.

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} (1/2)(e^{-2t}+1) & (1/2)(e^{-2t}-1) \\ (1/2)(e^{-2t}-1) & (1/2)(e^{-2t}+1) \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} + \begin{bmatrix} 25/2 \\ 25/2 \end{bmatrix}$$

where $(s_1(0), s_2(0))$ is the initial point at time 0. The eigenvalues expressed in the solution are -2 in e^{-2t} and 0 in $e^{0t} = 1$. For example, if $(s_1(0), s_2(0)) = (10, 4)$, then $(s_1(\infty), s_2(\infty)) = (31/2, 19/2)$ where the sum of the contributions is 25, and hence (31/2, 19/2) is a Nash equilibrium. Figure 2 (a) shows this case. Player 1's best response curve is $b - a - s_2$ and 2's best response curve is $a - b - s_1$. The intersection of these curves is the line segment *a-b* that is the set of Nash equilibria. As the phase diagram shows, the system is stable. Let $c = (s_1(t), s_2(t))$ at time t. Since $(-s_2(t) + 25) - s_1(t) > 0$, c must go upward as the arrow shows. Applying the same procedure to every possible initial point, I find that the system is stable. That is,

Proposition 4. *If the utility function is QL1, the system* $\dot{\mathbf{s}}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{s}(t) + \mathbf{k}$ *is asymptotically stable at Nash equilibria.*

Although the system of the VCM with (3) is stable in the sense that any initial point approaches some equilibrium point, it is vulnerable to parameter changes. Let us expand the class of QL1 utility functions in the following manner:

(QL1′) $v(s_i, s_{-i}) = w - s_i + z(s_i) + t(s_i + s_{-i})$.

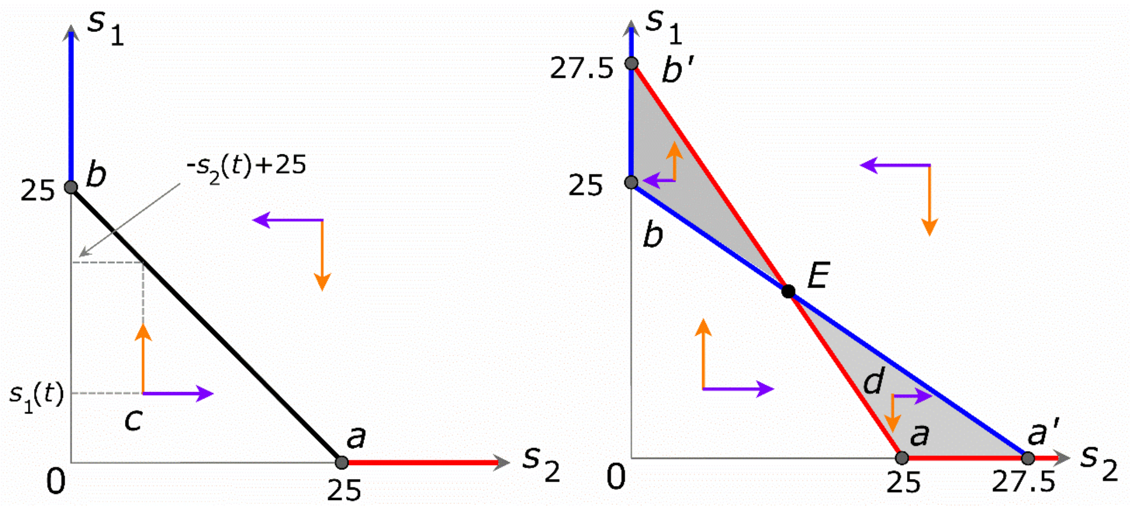
The QL1′ utility function is linear with respect to private consumption, but it is not linear with respect to *i*'s own contribution (i.e., the warm-glow part, $z(s_i)$) and the public good. Let us consider a QL1′ example in which both the warm-glow and the public good parts are quadratic.

$$v(s_i, s_{-i}) = w - s_i + gs_i^2 + 2(s_i + s_{-i}) - \frac{1}{50}(s_i + s_{-i})^2$$

[12] See Saijo and Kobayashi (2014).

where $w - s_i$ is the consumption of the private good, gs_i^2 is the warm-glow term, and $2(s_i + s_{-i}) - (1/50)(s_i + s_{-i})^2$ is the public good term. If g = 0, the function is QL1, and if g > 0, there is a warm-glow effect, i.e., more contribution is better for the contributor. Differentiating with respect to s_i, and then solving the first order condition, we have

$$s_i = \frac{25}{1-50g} - \frac{1}{1-50g} s_{-i}.$$



(a) Multiple Nash equilibria (b) Unstable Nash equilibrium

Figure 2. Phase diagrams.

Consider the case where both players whose g value is zero originally change their mind and let g = 1/550. Then the slope of the best response function is -1.1(=1/(10/11)). This subtle parameter change causes drastic changes in the system. Figure 2(b) shows the change. Point b on player 1's best response curve now jumps to b', while point a on player 2's best response curve shifts to a'. This change makes the Nash equilibrium *unique* although the set of Nash equilibria in Figure 2 (a) is the line segment a-b. This change gives rise to new areas such as E-a-a' and E-b-b' that are dark areas in the figure. Consider d. Since it is above player 1's best response curve, it should go down, and since it is below player 2's best response curve, it should go up, i.e., go to the right. That is, d will move away from the Nash equilibrium E. In fact, E is the saddle point of the phase diagram. That is, both Nash equilibrium and stability properties changed due to a small parameter change.

If both players change the slopes of their best response functions from -1 to -1.1, the eigenvalues of **A-I** are -2.1 and 0.1. The solution to the system of simultaneous linear differential equations is

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} (1/2)(e^{-2.1t}+e^{0.1t}) & (1/2)(e^{-2.1t}-e^{0.1t}) \\ (1/2)(e^{-2.1t}-e^{0.1t}) & (1/2)(e^{-2.1t}+e^{0.1t}) \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} + \begin{bmatrix} 275/21 \\ 275/21 \end{bmatrix}.$$

Since one of the eigenvalues is greater than 0, the system is not stable. In summary, if the utility function is QL1′, the players' warm-glow utility makes the system unstable.

On the other hand, if g = -1/550 < 0, i.e., if the utility function has a *cold*-glow component, the slope of the best response function is -11/12 > -1, and hence the Nash equilibrium is unique and the system is stable.

This example has an important implication for research on VCM using the class of utility functions that includes QL1 functions. We say that the system of differential equations is *structurally stable* if the phase diagram is invariant to perturbations of parameters of utility functions.[13] As the warm-glow example shows, we have the following important proposition.

Proposition 5. *If the utility function is QL1′, the system* $\dot{\mathbf{s}}(t) = [\mathbf{A} - \mathbf{I}]\mathbf{s}(t) + \mathbf{k}$ *is structurally unstable.*

That is, the system of simultaneous differential equations in the class of QL1′ functions has a bifurcation point when the parameter g is zero. In this sense, the VCM might not be an appropriate device for providing public goods since its stability is not guaranteed. Furthermore, as Anderson, Goeree, and Holt (1998) pointed out, players make errors in decisions, and hence they change the shape of their best response functions.

4. Previous Experimental Results

Table 1 summarizes some of the previous experimental results obtained with nonlinear utility functions. Each paper has its own purposes. These include checking the crowding out property, the endowment effect, the punishment effect, the altruism effect, social-tie formation, and so on, but I focus upon the very basic part of their experiments, namely average contributions. Table 2 summarizes the stability properties of the VCM.

First, consider the cases when the utility function is QL1. Isaac, McCue, and Plott (1985) observed "pulses" of individual contribution patterns with ten subjects of two types, although subjects in their model have dominant strategies. Pulsing behavior has been observed in linear environments using a different perspective. For example, Isaac, Walker, and Williams (1994) reported, "(s)uch 'pulsing' behavior could be interpreted as attempts to influence others' allocations through signalling" (p.16). Although it is uncertain whether this behavior might aggravate instability in nonlinear environments, surprisingly, no "pulses" were mentioned in experiments with QL1 environments. As Sefton and Steinberg (1996) pointed out, contributions

[13] According Hirsch and Smale (1974), "when does the phase portrait itself persist under perturbations of the vector field? This is the problem of structural stability" (p.304). As for mathematical rigor on structural stability, see chapter 1 in Devaney (1989) and chapter 16 in Hirsch and Smale (1974).

in the QL1 environment (the top row in Table 1) are more variable than in the QL2 environment (the second row from the bottom in Table 1). Laury and Holt (2008) attributed this to the multiplicity of Nash equilibria, but not to the instability of the system. The average contributions in the experimental results of Sefton and Steinberg (1996), Isaac and Walker (1998), and Laury, Walker, and Williams (1999) are far away from the Nash equilibrium amounts.

Table 1. Parameters and average contributions in nonlinear experiments

	Util. fun.	n	a	$\lvert a(n-1)\rvert$	w	(Symmetric) Nash Eq.	Average Cont.
Sefton-Steinberg (96)[*1]	QL1	4			8	2	4.42
Isaac-Walker (98)					62	12	27.225
						31	27.4
						50	44.574
Laury-Walker-Williams (99)		5			125s[*2]	20	62.47
					200s[*2]		71.28
					125d[*3]		43.31
					200d[*3]		44.06
Andreoni (93)	CD	3	-1/2	1	7	3	2.78
Chan-Mestelman-Moir-Muller (96)				1	20,20,20	5,5,5	5.3
					18,18,24	5,5,5	5.1
Cason-Saijo-Yamato (02)		2	-0.47	0.47	24	8	7.37[*3]
Sutter-Weck-Hannemann (04)		3	-1/2	1	7	3	3.18
Sefton-Steinberg (96)[*1]	QL2	4	0		8	2	3.97
Keser (96)					20	7	10.29
Maurice-Rouaix-Willinger (13)					20	7	8.58-8.9
Cason-Gangadharan (14)					12	3	5.69

[*1] Involving stranger matching.
[*2] w is at either 125 or 200 and summary information on the payoff structure is provided.
[*3] w is at either 125 or 200 and detailed information on the payoff structure is provided.
[*3] The average of 5 different locations in the USA and Japan and involving stranger matching.

The pattern of average contribution relative to the amount of endowment in Isaac and Walker (1998) is quite interesting. They observed that the sum of contributions is above (below) the symmetric Nash equilibrium when the equilibrium amount is low (high) relative to the total endowment. If the equilibrium is at the midpoint of the total endowment, the data are also around the middle. Anderson, Goeree, and Holt (1998) named this phenomenon the *sandwich* property, since experimental data are sandwiched by Nash equilibria, and provided a theoretical interpretation of the property using quantal response equilibrium analysis. Players make mistakes around the Nash equilibrium; their errors are normally distributed, but the distribution must be truncated since contributions cannot be negative. If the Nash equilibrium is below the midpoint of the endowment, the truncated distribution must have relatively large weight on the area where contributions exceed the Nash equilibrium amount. Hence, the contribution data must exceed the Nash equilibrium contribution. However, in order for this equilibrium analysis to be valid, the system must be stable.

Although individual data is not provided in Laury, Walker, and Williams (1999), the graphs of 90% confidence intervals of contributions show that they are relatively large (Figure 5 in Laury, Walker, and Williams (1999)). The contributions do not die down towards the ends of periods, and one of the frequency graphs for contributions is U-shaped (Figure 6 in Laury, Walker, and Williams (1999)), i.e., high and low contributions are more frequent relative to intermediate contributions.

Second, consider the cases when the utility function is CD. Unlike for the QL1 case, the Nash equilibrium is unique and $|a(n-1)|=1$ in Andreoni (1993), Chan, Mestelman, Moir, and Muller (1996) and Sutter and Weck-Hannemann (2004). That is, it is the boundary of the stability condition for the difference equation case, and hence there are many interior cycles (please see Table 2). Since $|a(n-1)|=0.47<1$ in Cason, Saijo, and Yamato (2002), the system is stable in the difference equation case. As for the differential equation case, all systems are always stable. In both cases, the instability of CD is weaker than that of QL1.

Table 2. Stability properties of previous experiments

	U.f.	Difference eq.	Differential eq.
Sefton-Steinberg (96), Isaac-Walker (98), Laury-Walker-Williams (99)	QL1	Unstable; boundary cycles; no interior cycles	Stable, but structurally unstable
Andreoni (93), Chan-Mestelman-Moir-Muller (96), Sutter-Weck-Hannemann (04)	CD	Unstable; interior cycles	Stable
Cason-Saijo-Yamato (02)	CD	Stable; no cycles	Stable
Sefton-Steinberg (96), Keser (96), Falkinger-Fehr-Gachter-Winter-Ebmer (00), van Dijk-Sonnemans-van Winden (02), Uler (11), Maurice-Rouaix-Willinger (13), Cason-Gangadharan (14)	QL2	Stable; no cycles	Stable

The average contributions in each CD case are close to the Nash equilibrium contribution even though the Nash equilibrium is below the midpoint of the endowment. That is, the sandwich property is not observed in the CD case.

Finally, consider the case when the utility function is QL2. The system is stable in both the difference and differential cases, but the average contributions are more than what the dominant strategies dictate.[14] That is, the contribution pattern in the QL2 case is similar to that shown in the linear case in the provision of public goods. As Cason and Gangadharan (2014) pointed out, this environment has *something* that triggers *altruism, reciprocity, social-ties,* or some other factors.

5. Concluding Remarks

I showed that the VCM system in a QL1 environment is unstable under simultaneous difference equations and structurally unstable under simultaneous differential equations. The VCM in a CD environment is always stable under simultaneous differential equations, while it is also stable under simultaneous difference equations if $|a(n-1)|<1$.

Although experimental economists have been emphasizing *internal* validity of the VCM system with linear utility functions, they have paid little attention to *external* validity.[15] This system is an excellent test bed for understanding many aspects of human nature, but it does not provide experimental basis for the provision of public goods if utility functions are not linear. If utility functions are QL1, researchers must deal with the instability of the system, and hence the VCM might not be an appropriate mechanism for analyzing public good provision. Thus, an open research question is to find what types of utility functions people really have, using experimental methods including field research.

Furthermore, a more fundamental question is whether the VCM is the central focus in public good provision. In the seventies, when the research on preference revelation on the provision of public goods was at its peak, Johansen (1977) criticized this line of research saying "there seems to be little empirical evidence which shows the importance of the preference revelation issue" (p.147) arguing "a strategy which consists in playing down one's preferences for a public good in order to get a lower share in the costs of providing the good is not likely to succeed in an open political decision-making process involving elected representatives" (p.147). The same critique might apply to the VCM. Although I do not have any basis for this opinion, the instability of the VCM could be one of the reasons why public goods have been provided through political decision process in representative democracies. Thus, pursing external validity is also an important issue in VCM research.

There are also many unexplored areas within the realm of internal validity. As Table 1 shows, average contributions can be more or less than the equilibrium amounts dictated by

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[14] van Dijk et al. (2002) did not provide the specific data in their paper, but the contributions are about 30%–270% higher than dominant strategy contributions judging from their figures. Falkinger, Fehr, Gachter, and Winter-Ebmer (2000) and Uler (2011) also observed higher contributions than dictated by the dominant strategies. The contributions decreased toward the end of the periods just as for the linear utility function case. Maurice, Rouaix, and Willinger (2013) found mixed results. One treatment showed over-contribution and the other did not.

[15] See Guala (2005) for internal and external validity.

Nash equilibrium or dominant strategies. Of course, this can be explained by individual player characteristics such as altruism, equity, warm-glow, and so on, but the properties of the system itself including stability or instability might affect the subjects' behavior. For example, Saijo (2014) explained the sandwich property in Isaac and Walker (1998) using the instability of the system (instead of errors). The instability-based explanation is compatible with the data of Chan, Mestelman, Moir, and Muller (1996), and others, on CD utility functions.

Although I employed a simple best response approach, there could be many other ways to incorporate dynamics; these include responses to information on strategies used in several preceding periods, learning, and so on. Furthermore, characteristics of human nature such as altruism, reciprocity, social-tie formation, and social norms, for which evidence has been found over the last few decades, might play an important role in alleviating or aggravating instability.

From Table 1, we can see that many questions remain to be explored. For example, no experiment with $|a(n-1)|>1$ has been done when the number of players is relatively large to understand the stability property in a CD environment. That is, as Isaac, McCue, and Plott (1985) observed, the number of players could be an important factor influencing VCM stability with nonlinear utility functions. Ostrom (2006) summarized experimental results for common-pool resources, and found "some unexplained pulsing behavior" (p.150) of each appropriator's labor inputs.[16] The number of players Ostrom mentioned is 8, while the cases considered in Table 1 have between 2 and 5 players. That is, the number of players chosen so far could be around the boundary of stability and instability. Results that are valid with small numbers of players might not hold with a relatively large number of players. Another possibility is that some behavioral factors, possibly some learning processes, might counterbalance instability in the VCM with nonlinear utility functions and a small number of players.

Another important agenda is the location of the Nash equilibrium. The sandwich property was not observed in CD environments when the equilibrium contribution amounts were below the midpoint of the endowments. However, experiments where the Nash equilibrium amount is above the midpoint of the endowment have not been performed, even in QL2 environments.

Some devices such as punishment, reward, and/or communication might affect the performance of the VCM with nonlinear utility functions. Cason and Gangadharan (2014) reported that peer punishment works reasonably well, though not quite as well as a linear utility function. They used a QL2 utility function, which makes the system stable, but the effectiveness of devices such as peer punishment has not yet been explored in unstable environments, such as QL1 and CD.

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[16] Saijo and Kobayashi (2014) explained pulsing behavior using best response dynamics.

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