Social Design Engineering Series *SDES-2014-13*

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24th October, 2014

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A theory of sanctions: Objectives, degree of heterogeneity, and growth potential matter for optimal use of carrot or stick

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October 24, 2014

Abstract

While most economic literature on punishment and reward follows an experimental study on a symmetric version of a public goods game, we theoretically study sanction institutions by focusing on an asymmetric public goods game. Using a model for a private-value all-pay auction, we find that (1) the reward (punishment) is more effective to motivate people with greater (less) ability than median ability, (2) to improve the total effort, the reward (punishment) is better for more (less) heterogeneous people, and (3) reward tends to be optimal in the long run under the dynamics of group diversity change caused by enforced sanctions.

C72, D44, H41. keywords: public goods provision, all-pay auction, heterogeneity, punishment, reward; growth potential

We start by considering the following two lines of questioning.

- First, assuming that the employees of each company have the same capability, which company would have better productivity—a company that motivates employees through a system in which top performers are promoted (promotion-based incentive) or a company that motivates employees through a system in which bottom performers are demoted (demotion-based incentive)? Would the answer differ by type of work?

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- Second, as a mechanism to provide public goods (e.g., security maintenance, regulatory compliance, and carbon dioxide emissions reduction), which is more effective—rewarding individuals who contribute the most or punishing those who contribute the least? Would the answer change based on the nature of the public goods or the characteristics of people who provide the goods?

What these questions have in common is the fact that motivation through an external incentive is considered a solution because high performance, contribution, and efforts are necessary for the organization and society, even though they are costs for those who provide them. This study focuses on the most fundamental types of such incentives—praise the top and curse the bottom. We believe two different types of incentives—pulling from the top and pushing from the bottom—are required and can, in fact, be explained from three perspectives: difference in the objective function of a designer, level of heterogeneity within the population, and potential for ability–growth based on effort. Thus, this study aims to derive general rules for choosing reward or punishment based on an examination of these three points.[1]

We assume a scenario in which a given number of individuals simultaneously choose an effort level, input amount, or performance (hereinafter referred to as "effort level") and they receive a punishment or reward accordingly. Although the effort level is measured objectively, there is uncertainty about the type of people. The parameter related to the cost associated with the effort varies by individual and we assume that this is private information. The parameter represents the marginal cost of the level of effort. It follows specific common priors, and this knowledge is shared. The top performer is rewarded under a condition in which there is a reward, while the bottom performer is punished under a condition in which there is a punishment.

Here, we describe previous studies. Reward and punishment are studied in a wide range of fields.[2] Experimental research on public goods games is the field that studies reward versus punishment the most; there are numerous experimental studies that examine which of the two is more effective in promoting the provision of public goods. For example, Walker and Halloran (2004) compare the effect of sanctions in the context of a one-off event, Sefton, Shupp and Walker (2007) and Rand, Dreber, Ellingsen, Fudenberg and Nowak (2009) compare under repeti-

[1] Social psychology takes note of social mobility and studies this type of question. Examples include Wang and Leung (2010) and Wang, Leung, See and Gao (2011).

[2] Punishment and reward are studied in several disciplines, including management (Trevino, 1992), criminology (Sherman, 1993), legal science (Wittman, 1984; De Geest and Dari-Mattiacci, 2013), social psychology (Gürerk, Irlenbusch and Rockenbach, 2009), evolutionary biology (Sigmund, Hauert and Nowak, 2001; Herold, 2012), and even neuroscience (Frank, Seeberger and O'Reilly, 2004).

tive conditions (the former shuffle the identities of members and the latter fix their identities), Sutter, Haigner and Kocher (2010) and Gürerk, Irlenbusch and Rockenbach (2006) conduct experiments in which participants themselves were allowed to choose the system, and Nikiforakis and Mitchell (2013) analyze the conditions in which subjects acting as a third party determine reward and punishment. The results on the effect of reward and punishment suggested by these experimental studies vary. The reason for the mixed results lies in the fact that the subjects themselves, in addition to providing public goods, made decisions on sanctions. In other words, the subjects had to predict subsequent sanction decisions by others when making decisions on public goods (and it seems their predictions often turn out to be incorrect). Therefore, it seems that the results vary by experiment conditions, such as the number of repetitions and whether the identities of partners are shuffled or not. There are studies, such as Kamijo, Nihonsugi, Takeuchi and Funaki (2014), and Putterman, Tyran and Kamei (2011), that turn sanctions into a system or rule; however, they study only punishment systems—they do not examine reward systems. In other words, the analyses of reward and punishment systems from the studies on public goods games are based primarily on experimental studies. Almost no theoretical analysis—particularly that which takes asymmetry and uncertainty into account—has been performed.

Reward and punishment as incentive within an organization is related to a field of study known as contest theory. Among these studies, those that are better known include the all-pay auction studies in complete information setting (Barut and Kovenock, 1998; Siegel, 2009) and incomplete information setting (Glazer and Hassin, 1988; Moldovanu and Sela, 2001). For example, Moldovanu and Sela (2001) use a model that is almost identical to the model in our study and conduct a study on optimal reward allocation. Specifically, Moldovanu and Sela (2001) consider what is optimal in terms of maximizing the amount of average effort when allocating a fixed reward according to the ranking of the effort level. Similar problems are considered in Barut and Kovenock (1998) and Glazer and Hassin (1988); the common finding among these studies is that the "winner take all" approach in which the top-ranked individual is given the entire reward becomes the optimal allocation under a moderate condition. Brookins and Ryvkin (2014) examine the theoretical prediction of the contest theory in a laboratory setting.

Some recent studies on all-pay auction focus on prize and punishment. Moldovanu, Sela and Shi (2012) extend Moldovanu and Sela (2001) to a case in which a negative value of "reward" is allowed to be assigned by players when considering the problem of optimal allocation of reward. They show that under a certain condition, the optimal incentive is either top-rewarding or bottom-punishing. Since Moldovanu and Sela (2001) clarifies the two most remarkable incentives from a wide class of reward allocations, we initially focus on these two and explore them in detail using a more generalized model. Thomas and Wang (2013) analyze the carrot and stick in an environment in which contestants endogenously join the race. While the use of the carrot in their study has intrinsic limitations due to the entry decision (and this is one important aspect of the use of punishment), our study does not pose such a restriction on punishment use.

Our study makes the following three contributions. First, we derive an equilibrium strategy for each condition under a reward system and a punishment system, and by comparing these, we are able to understand the fundamental characteristics of, or differences in, motivations by reward and punishment. Reward and punishment differ in who begins putting in more effort. Specifically, a reward has an effect of making highly capable individuals work harder and making less capable individuals slack off. On the other hand, a punishment has an effect of making less capable individuals put in more effort, although its effect on highly capable individuals is limited. To explain this from the standpoint of marginal effect (in terms of the slope of the equilibrium strategy), individuals with median or higher capabilities are effectively motivated by reward (reward is larger than the slope of the equilibrium strategy) and individuals with less than median capabilities are more effectively motivated by punishment. Based on this, we see that a reward is better when the target is to improve the performance of top performers in a group while a punishment is better when the target is to improve the performance of bottom performers.

Second, based on the abovementioned equilibrium strategies, we examine the optimal use of punishment or reward varied by degree of heterogeneity within a group. We assume that the objective of the designer is to increase the average amount of effort. The analysis results show that reward is effective for a group with a large variance in capability while punishment is effective for a group with a small variance in capability. To explain this intuitively, the increased effort driven by punishment in turn affects highly capable individuals in a group with a small disparity in capability so that the punishment tends to become effective. In contrast, the effect of punishment on increasing the bottom effort in a group with a large gap in capability does not really reach high performers; therefore, promoting efforts among top performers using reward results in an increase in the amount of effort for the entire population. These theoretical results suggest that reward is effective for a group of people with different, multiple backgrounds while punishment is effective for a homogeneous group. To put these theoretical results another way, they mean that while reward is better for a type of work in which there is a large qualification gap for the task among individuals, punishment is more effective for a type of work in which there is a minimal qualification gap among individuals.

This finding on the proper usage of carrot and stick based on degree of heterogeneity of capability has not been pointed out in previous studies. Under the contest theory, it is known that increased diversity reduces the total sum of efforts (Schotter and Weigelt, 1992; Gradstein, 1995). This is because the presence of highly capable individuals demotivates less capable individuals and, as a result, those less capable individuals reduce their effort level, causing highly capable individuals to choose a half-hearted effort level. Because of this type of discouragement effect, the exclusion principle, which states that the total sum of efforts increases by excluding highly capable individuals, holds (Baye, Kovenock and Vries, 1993). However, these studies have not discussed the relationship between the carrot and stick. Moldovanu, Sela and Shi (2012) is the first study that refers to the relations between the distribution of capability and the optimal use of the carrot and stick, but their analysis is motivated more from the mathematical properties of the distribution function (convexity or concavity) and is unrelated to the extent of the capability gap in a focused group. Moreover, although there are experimental studies on the heterogeneity of players (endowments and cost) in the public goods game, including Buckley and Croson (2006), Chan, Mestelman, Moir and Muller (1996), and Nitta (2014) for endowment heterogeneity and Fisher, Issac, Schatzberg and Walker (1995), Tan (2008), and Fellner, Iida, Kröger and Seki (2011) for provision cost heterogeneity, there is no study that compares reward and punishment.

Third, we examine the interaction between external incentives and the change in diversity by hypothesizing a condition in which the capability of people develops according to their effort level. The key point is that even if an appropriate incentive (reward or punishment) is laid out initially, what is considered "an appropriate incentive" may change under a new condition when the capability distribution changes because of development. What we find is that, in the long run, a condition in which the reward is best suited tends to become established under an environment wherein people can develop their capabilities. Specifically, once the condition reaches a point at which the reward is best suited, the reward would always dominate in the subsequent distribution in which a reward or punishment is implemented (coherent reward dominance). Meanwhile, the environment in which a punishment is effective can be turned over (i.e., the reward could become dominant in the subsequent period's distribution even if it was after a punishment was implemented).

The rest of this paper is organized as follows. The next section describes the basic model based on all-pay auction and defines the reward system and the punishment system. Section 2 derives the equilibrium strategy for each system and, by comparing the equilibrium strategies, examines the merits of the effect of effort promotion by each system. Section 3 explains the relationship between the degree of heterogeneity of the group members (the degree of capability gap) and the optimal system. Section 4 examines the consequences for this optimal system by hypothesizing a dynamic environment in which the capability of each individual develops according to the amount of effort. Section 5 is the conclusion.

1 Basic model

1.1 General setup

The general setup follows the model of all-pay auction by Moldovanu and Sela (2001). Consider an effort selection game among n players. Let $N = \{1, 2, ..., n\}$ with $n \geqq 2$ be a set of participants of the game. Each i makes an effort $x_i \in [0, \infty]$. An effort can be interpreted as performance if it is in the context of workers' decisions in an organization, and as contribution to a public project if it is in the context of a public provision game. An effort incurs a disutility or cost of x_i/θ_i, where $\theta_i \in (\theta_L, \theta_H)$ is the cost parameter for i. Thus, the cost function of an effort is the product of the individual specific component (preference, wealth, ability, skill, and so on) and the selected provision component. A player with a high θ can make high effort by small cost.

In an effort selection game without any sanction institution, players choose their efforts simultaneously. Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in X^n$ be a profile of efforts. The payoff of player i choosing x_i is

$$-\frac{x_i}{\theta_i} + U(\mathbf{x}_{-i}),$$

where $U(\mathbf{x}_{-i}) \geq 0$ is the utility generated from others' behavior.[3] However, since this part does not affect the results of our further analysis, we omit this hereafter.

1.2 Punishment and reward

Let $N^{max}(\mathbf{x})$ and $N^{min}(\mathbf{x})$ be the number of highest performers and lowest performers in $\mathbf{x}$, respectively. Let $I_i^{max}(\mathbf{x})$ be i's maximum indicator function with value 1 if x_i is highest at $\mathbf{x}$ and 0 otherwise. Similarly, let $I_i^{min}(\mathbf{x})$ be i's minimum indicator function with value 1 if x_i is lowest at $\mathbf{x}$ and 0 otherwise.

Let $P > 0$ be a fixed amount of punishment. In a punishment institution, the payoff of player i in profile $\mathbf{x}$ is

$$v_i^p(\mathbf{x}) = -\frac{x_i}{\theta_i} - I_i^{min}(\mathbf{x})\frac{P}{N^{min}(\mathbf{x})}$$

[3] In the context of public goods provision, U is the utility from a public project that depends on the contribution profile of others. This implies that the utility gain from a public project by a worker's own contribution is reflected in the cost function x_i/θ_i. Thus, if this is a linear public goods provision game, $1 - 1/\theta_i$ can be seen as a marginal per capita return from a public good.

Thus, we assume that the punishment is directed toward the worst performer. If there is a tie, the amount of sanction is divided among the tied members, or one among the tied members is selected randomly to receive the sanction.

Let $R > 0$ be the amount of reward. Similarly, the payoff of player i in a reward institution is

$$v_i^r(\mathbf{x}) = -\frac{x_i}{\theta_i} + I_i^{max}(\mathbf{x})\frac{R}{N^{max}(\mathbf{x})}$$

The best performer gains a reward R, and if there is a tie, the reward is divided among the tied members or one member is chosen randomly and gains the entire reward amount.

We should clarify the differences in this study from Moldovanu and Sela (2001) and Moldovanu, Sela and Shi (2012). Moldovanu and Sela (2001) consider the class of reward allocation that adds up to the fixed M. Thus, in their framework, top rewarding is the form in which the top obtains M and others obtain nothing, and bottom punishment is the form in which the top $n-1$ obtains $M/(n-1)$ and there is nothing for the bottom. In contrast, we formulate the punishment as one that sanctions the bottom by $-M$ and nothing for the others. In this meaning, the analysis of Moldovanu and Sela (2001) underestimates the punishment institution compared with our study. Moldovanu, Sela and Shi (2012) extend the previous work to allow negative sanctions and to analyze the optimal combination of carrot and stick. They explore the optimal sanctions under the restriction that the domain of F is $[0, 1]$ and F satisfies the increasing failure rate (IFR) condition. The study shows that either top rewarding or bottom punishment becomes the optimal incentive among the class of incentives in which positive or negative values are assigned to participants based on the order of players' efforts. Thus, the analysis of Moldovanu, Sela and Shi (2012) gives reason to focus on top rewarding and bottom punishment.[4] To start, we explore the relationships between these two incentives and analyze new questions under a more moderate environment (the domain of F is not restricted to $[0, 1]$ and F need not satisfy the IFR condition).

2 Equilibrium analysis and median principle

An ability parameter of player i is private information to i. This is identically and independently distributed according to F. Let a distribution function F with its density f be increasing in its domain (θ_L, θ_H). Thus, an effort selection game with punishment or reward is an incomplete information game.

[4] In addition, Moldovanu, Sela and Shi (2012) mention that without the IFR condition, bottom punishment may not be the optimal form of punishment.

A strategy of player i in this game is a function that associates his realized type θ_i with effort x_i. Let β_i be the strategy of player i. We adopt the symmetric Bayesian Nash equilibrium $(\beta, \beta, ..., \beta)$, in which every player uses the same strategy β as a solution criterion in order to evaluate the performance of the sanction institutions. For an equilibrium strategy β, we assume that β is a continuous, differentiable, and increasing function with $\beta(\theta_L) = 0$. This is a standard assumption familiar to auction theorists.

We now explore the conditions that should be satisfied by β. First, we consider the punishment institution. Suppose $n-1$ players, except one player with type θ, follow the strategy β. Then, the expected payoff of the player when he chooses contribution x is as follows:

$$\Pi^p(x, \theta) = -\frac{x}{\theta} - P\left(1 - F(\beta^{-1}(x))\right)^{n-1}. \tag{1}$$

where E is an expectation operator.

Note that $F(\beta^{-1}(x))$ is the probability that a player following β chooses a performance less than x and $1 - F(\beta^{-1}(x))$ is the probability that a player chooses a performance more than x. Thus, $(F(\beta^{-1}(x)))^{n-1}$ is the probability that a player choosing x is the best performer among n players and $(1 - F(\beta^{-1}(x)))^{n-1}$ is the probability that a player choosing x is the worst performer among n players.

On differentiating the expected payoff by x, we have the following:

$$\frac{\partial \Pi^p}{\partial x} = -\frac{1}{\theta} - P(n-1)\left(1 - F(\beta^{-1}(x))\right)^{n-2}(-f(\beta^{-1}(x)))\frac{1}{\beta'(\beta^{-1}(x))}.$$

As β is the best response to others' β, the performance $x = \beta(\theta)$ must be a local maximum of the function $\Pi^p(., \theta)$, and thus, the above equation should be equal to 0 when evaluated at $x = \beta(\theta)$. On arranging the above equation, we have the following necessary condition of $\beta(\theta)$ to constitute equilibrium for the punishment institution:

$$\beta'(\theta) = P(n-1)(1 - F(\theta))^{n-2}\,\theta f(\theta).$$

On integrating this equation by θ with the condition $\beta(\theta_L) = 0$, we have the following:

$$\beta^p(\theta) = P(n-1)\int_{\theta_L}^{\theta}(1 - F(z))^{n-2}\,z f(z) dz. \tag{2}$$

Thus, we have a candidate for the equilibrium strategy of the punishment institution from the above equation.

Next, we explore the necessary conditions of β in the reward institution. Suppose $n-1$ players, except one player with type θ, follow the strategy β. Then, the

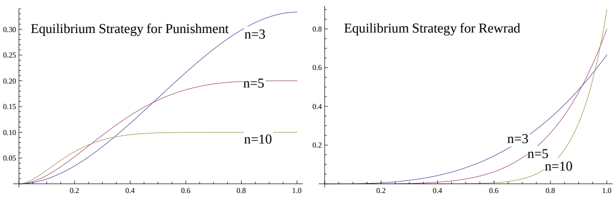

Figure 1: The equilibrium strategies for sanction institutions. The left panel corresponds to the equilibrium strategy for the punishment institution and the right panel corresponds to the equilibrium strategy for the reward institution. $(\theta_L, \theta_H) = (0, 1)$, $F(z) = z$, and $P = R = 1$.

expected payoff of a player who chooses performance x is as follows:

$$\Pi^r(x, \theta) = -\frac{x_i}{\theta_i} + R\left(F(\beta^{-1}(x))\right)^{n-1}. \tag{3}$$

By a similar procedure, we obtain the candidate for the equilibrium strategy of the reward institution as follows:

$$\beta^r(\theta) = R(n-1)\int_{\theta_L}^{\theta} (F(z))^{n-2}\, z f(z) dz. \tag{4}$$

It is easy to check that β^p and β^r are Bayesian Nash equilibrium strategies for punishment and reward institutions, respectively.

Figure 1 shows the equilibrium strategies for punishment and reward institutions for different group sizes. From this, we easily find that the effort-improving effects of these two incentives are totally different in "the person who is more responsive to the incentives." In the reward institution, a highly able player is strongly motivated and makes a very high effort, but the effort-improving effect on a person with low ability is limited. This is consistent with the finding from all-pay auction known as the discouragement effect, in which the more highly able person discourages a less able person(Schotter and Weigelt, 1992; Gradstein, 1995). The converse holds for the punishment institution; a person with low ability is strongly motivated while the effect on a highly able person is limited. Another important difference is that, while in punishment the difference in effort between low and high ability players is moderate, there is considerable difference in the effort between them in the reward institution.

Comparing β^p and β^r under the condition $P = R$, we find a useful insight regarding punishment and reward. Let θ^{med} be the parameter satisfying $F(\theta^{med}) = 1/2$, that is, the median of F.

Theorem 1 (Median principle)**.** *Assume $R = P$. The following hold:*

(1) for any $\theta \leqq \theta^{med}$, $\beta^p(\theta) > \beta^r(\theta)$

(2) for any $\theta < \theta^{med}$, $(\beta^p)'(\theta) > (\beta^r)'(\theta)$

(3) for any $\theta = \theta^{med}$, $(\beta^p)'(\theta) = (\beta^r)'(\theta)$

(4) for any $\theta > \theta^{med}$, $(\beta^p)'(\theta) < (\beta^r)'(\theta)$

(5) there exists $\hat{\theta} < \theta_H$ such that for any $\theta > \hat{\theta}$, $\beta^p(\theta) < \beta^r(\theta)$

Proof. (1) to (4) are obvious from Eqs (2) and (4). We prove (5) of the theorem because when $P = R$,

$$\beta^r(\theta_H) - \beta^p(\theta_H) = R(n-1)\int_{\theta_L}^{\theta_H} \left((F(z))^{n-2} - (1-F(z))^{n-2}\right) z f(z) dz$$

$$= R\int_{\theta_L}^{\theta_H} \left((F(z))^{n-1} + (1-F(z))^{n-1}\right)' z dz$$

$$= R\left[\left((F(z))^{n-1} + (1-F(z))^{n-1}\right) z\right]_{\theta_L}^{\theta_H} - R\int_{\theta_L}^{\theta_H} \left((F(z))^{n-1} + (1-F(z))^{n-1}\right) dz$$

$$= R(\theta_H - \theta_L) - R\int_{\theta_L}^{\theta_H} \left((F(z))^{n-1} + (1-F(z))^{n-1}\right) dz$$

$$= R\int_{\theta_L}^{\theta_H} 1 dz - R\int_{\theta_L}^{\theta_H} \left((F(z))^{n-1} + (1-F(z))^{n-1}\right) dz$$

$$= R\int_{\theta_L}^{\theta_H} \left(1 - (F(z))^{n-1} - (1-F(z))^{n-1}\right) dz > 0.$$

This means that for θ close to θ_H, $\beta^r(\theta) > \beta^p(\theta)$. □

This indicates that median ability plays a key role in understanding the characteristics of the performance-improving effect of the two institutions. The first statement of this theorem states that less able players make more effort in punishment institutions than in reward institutions. Thus, a person with less ability than the median is more strongly motivated by the threat of punishment. In fact, the marginal increase in the efforts of less able players by punishment is greater than that by reward, indicating that for less able players, avoiding the stick is more effective than being shown the carrot ((2) of the theorem). This marginal incentive is reversed for more highly able players who are more motivated by the carrot ((4) of the theorem), but their effort level for the reward may be smaller than that for the punishment because of the cumulative pushing-up or chain-reaction effects of punishment, that is, pushing-up effects on less able players weakly contribute to the progress of more highly able players. However, the equilibrium contribution of very highly able players is higher with the reward than the punishment ((5) of the theorem).

3 Group diversity and the optimal use of sanctions

The designer's problem is to choose the better sanction institution from reward or punishment based on the designer's objective and group characteristic. The designer's goal is to increase the effort activity of all group members, and thus, to maximize the total sum of effort (or, average effort) among the group members. One reason to focus on the sum of effort is that this form of technology is the most common in the literature on contest theory and public goods provision games. Another is that this case is less obvious for the abovementioned problem, while the best shot technology and the weakest link cases are more obvious, based on our analysis in Section 2; the reward institution for the best shot and the punishment institution for the weakest link.[5]

We assume that $P = R = M$. The goal of this section is to demonstrate how the diversity of group members affects the optimal choice of the punishment and the reward. Thus, the designer has to use the two institutions based on the group's diversity.

For a distribution function F, let $\lambda(x)$ be a $F^{-1}(x)/x$ for $x \in (0, 1]$, which is the slope connecting the origin to $(x, F^{-1}(x))$. The following is our main result on the optimal use of the punishment and reward.

Theorem 2. *The following points hold:*

(i) If $\lambda(x) \geqq \lambda(1 - x)$ for all $x \in (0, 1/2)$ and strict inequality holds for some $x \in (0, 1/2)$, punishment is better than reward,

(ii) if $\lambda(x) \leqq \lambda(1 - x)$ for all $x \in (0, 1/2)$ and strict inequality holds for some $x \in (0, 1/2)$, reward is better than punishment, and

(iii) if $\lambda(x) = \lambda(1 - x)$ for all $x \in (0, 1/2)$, there is no difference between punishment and reward.

[5] This point is theoretically analyzed in a previous version of the paper (Kamijo, 2014). A remarkable finding in Kamijo (2014) is that contrary to intuition, the reward (punishment) can be better under the weakest-link (best-shot) technology if the heterogeneity (homogeneity) of group members is very strong. The terms "best-shot" and "weakest-link" are used in the context of the public goods game (Hirshleifer, 1983).

Proof. Let G_k^n denote the distribution function of the k-th order statistic for n independent random variable following F, and g_k^n be the density of G_k^n. Note that $G_n^n(z) = 1-(1-F(z))^n$, $g_n^n(z) = nf(z)(1-F(z))^{n-1}$, $G_1^n(z) = F(z)^n$, $g_1^n(z) = nf(z)F(z)^{n-1}$. Expected effort of one player under punishment and reward is:

$$E^p = E[\beta^p(\theta)] = \int_{\theta_L}^{\theta_H} \beta^p(\theta)f(\theta)d\theta = \int_{\theta_L}^{\theta_H}\left[M\int_{\theta_L}^{\theta} g_{n-1}^{n-1}(z)zdz\right]f(\theta)d\theta$$

and

$$E^r = E[\beta^r(\theta)] = \int_{\theta_L}^{\theta_H} \beta^r(\theta)f(\theta)d\theta = \int_{\theta_L}^{\theta_H}\left[M\int_{\theta_L}^{\theta} g_1^{n-1}(z)zdz\right]f(\theta)d\theta.,$$

respectively.

On interchanging the order of integration of E^p, we obtain the following:

$$E^p = M\int_{\theta_L}^{\theta_H}\left[\int_z^{\theta_H} f(\theta)zd\theta\right]g_{n-1}^{n-1}(z)zdz = M\int_{\theta_L}^{\theta_H}[1-F(z)]\,g_{n-1}^{n-1}(z)zdz,$$

which is equal to

$$M(n-1)\int_{\theta_L}^{\theta_H}[1-F(z)]\,f(z)(1-F(z))^{n-2}zdz.$$

Let $x = F(z)$, and we have

$$E^p = M(n-1)\int_0^1(1-x)(1-x)^{n-2}F^{-1}(x)dx.$$

By a similar calculation, we obtain the following:

$$E^r = M(n-1)\int_0^1(1-x)x^{n-2}F^{-1}(x)dx.$$

Thus, we obtain

$$\begin{aligned}E^p - E^r &= M(n-1)\int_0^1\left[(1-x)^{n-2}-x^{n-2}\right](1-x)F^{-1}(x)dx\\&= M(n-1)\int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](1-x)F^{-1}(x)dx\\&+M(n-1)\int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](x)F^{-1}(1-x)dx.\end{aligned}$$

Thus, we obtain

$$E^p - E^r = M(n-1)\int_0^{1/2}\left[(1-x)^{n-2} - x^{n-2}\right]\left[(1-x)F^{-1}(x) - (x)F^{-1}(1-x)\right]dx. \tag{5}$$

Thus, when $(1-x)F^{-1}(x) - (x)F^{-1}(1-x) \geqq 0$, $E^p - E^r \geqq 0$. Thus, the proof of (i) ends. From this, (ii) and (iii) are immediately proved. □

This theorem provides a sufficient condition for when punishment is better or worse than reward. The point is $\lambda(x)$. According to Theorem 2 (i), the punishment (reward) is better if $\lambda(x)$, the slope of the line connecting $(0,0)$ and $(x, F^{-1}(x))$ is smaller (greater) than $\lambda(1-x)$, as well as the slope of the line connecting $(0,0)$ and $(1-x, F^{-1}(1-x))$, for any $x \in (0, 1/2)$. Furthermore, this theorem implies that if $\lambda(x)$ is decreasing (increasing) in $(0,1]$, the punishment (reward) is better. For instance, let us consider the class of distribution function $F(\theta) = \theta^\alpha$ on $[0,1]$ with $\alpha > 0$. When $\alpha > 1$, $F^{-1}(x)/x = x^{(1/\alpha)-1}$ is decreasing, the punishment is better than the reward. When $\alpha < 1$, the converse holds, and thus, the reward is better than the punishment. More generally, if F is a concave (convex) function with $F(0) = 0$, $\lambda(x)$ is apparently increasing (decreasing), and thus, the reward (punishment) is better. Convexity or concavity of F is the condition that Moldovanu, Sela and Shi (2012) focus on in order to compare the top-rewarding and bottom-punishing incentives. As the next proposition shows, Theorem 2 can apply to the case in which F is neither convex nor concave.

To demonstrate how the diversity of group members affects the optimal institution, we restrict our attention to the class of symmetric distribution on $[0,1]$.

Proposition 1. *Assume that F is a symmetric distribution on* $[0,1]$.

(i) If $\lambda(x) \geqq 1$ for all $x \in (0,1/2]$ and strict inequality holds for some $x \in (0,1/2]$, punishment is better than reward,

(ii) if $\lambda(x) \leqq 1$ for all $x \in (0,1/2]$ and strict inequality holds for some $x \in (0,1/2]$, reward is better than punishment, and

(iii) if $\lambda(x) = 1$ for all $x \in (0,1/2]$ and strict inequality holds for some $x \in (0,1/2]$, there is no difference between punishment and reward.

Proof. It is obvious from Theorem 2 since $\lambda(1-x) = \frac{F^{-1}(1-x)}{1-x} = \frac{1-F^{-1}(x)}{1-x}$ when F is a symmetric distribution function on $[0,1]$. □

Note that when $\lambda(x) = 1$ for all x, F is a uniform distribution on $[0,1]$. F is less heterogeneous than a uniform distribution when $\lambda(x) \geqq 1$ for all $x \in (0,1/2]$, and more heterogeneous than a uniform distribution when $\lambda(x) \leqq 1$ for all $x \in$

$(0, 1/2]$. From this proposition, we know there is indifference about enhancing the sum of effort for reward and punishment if F is a uniform distribution. If F is less heterogeneous than the uniform distribution, punishment is better than reward. On the contrary, if F is more heterogeneous than the uniform distribution, reward is better than punishment. Thus, the diversity of group members matters for the optimal choice of punishment and reward.

Next, we investigate how the optimal institution changes if the abilities or skills of group members increase evenly. For $A > 0$ and a distribution function F on $[\theta_L, \theta_H]$, F_{+A} is defined by $F_{+A}(\theta) = F(\theta - A)$ for $\theta \in [\theta_L + A, \theta_H + A]$. F_{+A} is said to be a right parallel shift of F. Thus, F_{+A} is a distribution function after uniformly increasing the skill parameters of group members from an initial distribution F.

Proposition 2. *There exists some $A > 0$ such that under a distribution function F_{+A}, the punishment is better than the reward.*

Proof. Note that $F_{+A}^{-1}(x) = F^{-1}(x) + A$ for any x. From (5), the expected total sum in the punishment less than in the reward under F_{+A} is reduced to

$$M(n-1)\int_0^{1/2}\left[(1-x)^{n-2} - x^{n-2}\right]\left[(1-x)F^{-1}(x) - (x)F^{-1}(1-x)\right]dx$$

$$+AM(n-1)\int_0^{1/2}\left[(1-x)^{n-2} - x^{n-2}\right][1-2x]\,dx.$$

The first term is a finite number and the second term is always positive. Thus, for extremely large A, the equation above becomes positive. □

This proposition shows that as the lowest ability increases, punishment tends to be chosen as the optimal sanction. This is consistent with the diversity view, that less diversity implies punishment and more diversity implies reward. Fixing the shape of F, a large right parallel shift implies that the group becomes more homogeneous with higher average ability. Imagine a selected group of baseball players in some country. Their average skills are somewhat higher than the average of all baseball players in the country but their abilities are more homogeneous. In such a case, punishment can better improve their performance than reward, even though their average performance is quite high.

4 Interaction between sanctions and group diversity

In this section, we consider a situation in which people grow in their abilities or skills according to their efforts. Firms competing in a R&D race accumulate knowledge and technology irrespective of whether they win or lose that race; the effort of a student to obtain an educational opportunity improves his/her abilities; and employees' dedication to a task in an organization advances their fitness to that task. Overall, in many situations, people's abilities increase or decrease as a result of their efforts, which are affected by enforced incentives, that is, the carrot or stick.

We model the interaction between the enforced incentives and the change of group diversity by considering a dynamic situation in which in each period, people rationally respond to incentives and the environment; before the next period, their abilities vary according to their selected efforts and the new distribution function becomes common knowledge. We assume a "myopic rational player" who plays the equilibrium strategy of a one-shot game in each period but does not consider future events, such as the enforced incentives, his/her growth, and others' growth in the next periods. In addition, we assume a "large population and random matching" setting in which n strangers are picked from a large population (with distribution function F); they play the one-shot game and this event is repeated several times in each period without information feedback and update. These assumptions enable us to dispense with analyzing a complicated dynamic game. All we need is to analyze how the distribution function changes from the initial one by the enforced incentives.

Let g be an ability growth function, which implies a growth rate in his/her ability after choosing effort. Type θ player with a chosen effort b improves or worsens his/her ability to $\theta g(b)$ in the next period. We assume that g is a non-decreasing continuous function with $g(b) > 0$ for all $b \geqq 0$. Let $\beta(\theta)$ be an effort selection function (β^p if punishment is enforced, and β^r if reward is enforced) in some period and F be a distribution function of their abilities in that period. Then, after their ability growth, the distribution function in the next period is

$$F_2(\theta) = \text{Prob}\{\, x \in (\theta_L, \theta_H) \mid x \cdot g(\beta(x)) \leqq \theta \,\}$$

Put $h(x) := x \cdot g(\beta(x))$ and note that h is increasing. Then,

$$\begin{aligned} F_2(\theta) &= \text{Prob}\{x \in (\theta_L, \theta_H) \mid h(x) \leqq \theta \,\} \\ &= F(h^{-1}(\theta)) \end{aligned}$$

Therefore, we have, for any $y \in [0, 1]$,

$$\begin{aligned} F_2^{-1}(y) &= h(F^{-1}(y)) \\ &= F^{-1}(y)g(\beta(F^{-1}(y))) \end{aligned}$$

From the viewpoint of optimal use of punishment or reward, our concern is whether the optimal institution in some period remains optimal in the next period with a different distribution in people's abilities, that is, whether the optimal sanction institution is coherent in the next period. The next theorem answers this question.

Theorem 3 (Coherent Reward Dominance)**.** *If reward is better than or equal to punishment in some period t, reward is better than punishment in period $t+1$, regardless of whether punishment or reward is used in period t.*

Proof. By the assumption of the theorem, we have, from (5),

$$E_p^t - E_r^t =$$

$$M(n-1)\int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right]\left[(1-x)F^{-1}(x)-(x)F^{-1}(1-x)\right]dx \leqq 0 \tag{6}$$

in period t. Let β be an equilibrium performance function and $k(x) = g(\beta(F^{-1}(x)))$. Note that k is an increasing function. Then,

$$E_p^{t+1} - E_r^{t+1} =$$

$$M(n-1)\int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right]\left[(1-x)F^{-1}(x)k(x)-(x)F^{-1}(1-x)k(1-x)\right]$$

This must be negative because

$$\begin{aligned}
&\int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](1-x)F^{-1}(x)k(x)dx \\
&< \int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](1-x)F^{-1}(x)k(1/2)dx \\
&\leqq \int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](x)F^{-1}(1-x)k(1/2)dx \quad \text{(from (6))} \\
&< \int_0^{1/2}\left[(1-x)^{n-2}-x^{n-2}\right](x)F^{-1}(1-x)k(1-x)dx
\end{aligned}$$

□

This theorem shows that once the distribution becomes one in which the reward is optimal, the dominance of the reward institution is never reversed, even after the enforcement of the punishment. Thus, the reward institution is inherently dominant in the dynamics of distribution changes. Does the punishment institution have a similar property? The next corollary hints at the answer.

Corollary 1. *Suppose that an initial distribution is uniform in* $[0, 1]$*. Then, the reward institution is better than the punishment institution under an updated distribution in the next period, regardless of whether punishment or reward is used in the first period.*

Proof. This is obvious from Theorems 2 and 3. □

This corollary implies that there exists an initial distribution such that the punishment is better than the reward but the reward becomes better than the punishment in the next period after the punishment is enforced. The reason is simple. Let U_2 be an updated distribution function by the punishment from an initial uniform distribution function U on $[0, 1]$, and let E^p and E^r be the expected sum of the efforts under U_2 when the enforced sanction is the punishment or the reward, respectively. The corollary says that $E^r - E^p$ is strictly positive. Now, consider the initial distribution function F such that F is symmetric, very close to U, and more homogeneous than U. Then, from Proposition 3, the punishment is better than the reward under F. Let F_2 be the updated distribution function by the punishment from an initial distribution F. Because $E^r - E^p > 0$ and this is continuously changed by the infinitely small change of an initial distribution function, the reward is better than the punishment under F_2. Therefore, the punishment institution does not have the coherent dominance property.

5 Conclusion

This study analyzed the effects and characteristics of reward and punishment systems by using all-pay auction models operated by risk-neutral individuals who have private information on the capability of subjects. (1) By comparing the equilibrium strategy for each system, we confirmed that whereas the reward system is effective in significantly increasing the effort levels of highly capable individuals, the punishment system increases the capability of less capable individuals, which, in turn, gently promotes the effort level among highly capable individuals. (2) From the standpoint of the designer who aims to increase the sum of the effort, it became clear that the reward is effective when heterogeneity within the group is large (the gap is large) and the punishment is effective when the gap is small. (3) In terms of long-term effects that take into account the growth of people, the study demonstrated that while the reward is coherent by remaining effective once it becomes effective under a certain condition, the punishment does not have this type of coherency.

Based on the abovementioned three findings, the appropriate usage of reward and punishment incentives is summarized as follows. First, a punishment is better when the objective of implementing an incentive is that everyone is required to put out a certain level of effort or results (weakest-link), while a reward is better when it is sufficient if some individuals produce excellent results (best-shot). Thus, a penalty-type incentive is suitable to enforce regulations or maintain order while a reward-type incentive is suitable for such activities as research and art.

Furthermore, the capability gap within the group subject to the incentive is important. Generally speaking, the reward type works better for a group of individuals who have not been specially screened because the gap is probably large. In contrast, the punishment type could be effective for a group consisting of screened individuals. In terms of relationship with task, the punishment type is better for a kind of task that can be done by anyone because the capability gap would probably be small, while the reward type is better for a kind of task that can only be performed by some people. It is interesting that this implication conforms with the conclusion based on the perspective of justice (i.e., it is wrong to motivate people by punishment when it is known there are individuals who cannot fulfill the requirement; see De Geest and Dari-Mattiacci, 2013). Furthermore, the implication is in accordance with the conclusion by Wittman (1984) regarding the cost of implementing sanctions, that the incentive should be designed to minimize the number of individuals on whom the sanction is imposed.

Considering the growth potential associated with effort, the reward type would work better for the study of each subject in school education because the growth potential is high. On the other hand, the punishment type is recommended for specific matters, such as mastering a code of behavior or learning social norms, because the growth potential is probably low. Likewise, in a company, while the reward type (promotion) works better for technical and research positions in which workers have room to improve their expertise further, the punishment type (demotion) is appropriate for areas that are less technical or that level off quickly. It is noteworthy that in our framework, the situation in which only the bottom person is not promoted is identical to that in which only the bottom person receives the punishment, and thus, we view as a punishment type the promotion race wherein only bottom performers cannot escalate their careers while the others can.

We presented a variety of guidelines for proper use of the carrot and stick; however, to some extent, these results should be taken lightly. First, as has been demonstrated in such fields as behavioral economics, people tend to become so afraid of punishment that they put too much effort in attempting to avoid it. This not only suggests that our results understate the effectiveness of the punishment system but also implies that it is extremely difficult to predict the reactions of people towards punishment (Trevino, 1992). Therefore, opportunities to use punishment may naturally be limited. Second, our model ignores intrinsic motivation that people have for behavior. It is necessary to consider that extrinsic motivation, such as the carrot and stick, often ruin intrinsic motivation.[6] Third, although a technical issue, our assumption in deriving the equilibrium strategies, that high performers would always work harder, even under a punishment-type incentive, is largely responsible for the punishment type not having the characteristic of coherent dominance, as the reward type does. Thus, coherent dominance may also hold true for the punishment type if it is possible to configure the equilibrium by removing this assumption and recognizing that highly capable individuals slack off under punishment incentives. This point should be examined as a future theoretical development.

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[6]Discussions about an appropriate combination of intrinsic and extrinsic motivation are far beyond the objectives of this study.

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